E2:202 Random Processes **Oct. 23, 2020**

Tutorial 3: Random Variables, Random Vectors and Expectation

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In the earlier tutorials, we had seen discussions on when a function $X : \Omega \to \mathbb{R}$ (for a suitably defined $(\Omega, \mathcal{F}, \mathbb{P})$) is a random variable.
In this part, we look at functions of random variables and demonstrate a succint sufficient condition for these functions too to be random variables.

First, let us look at simple examples:
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given probability space, and let $X : \Omega \to \mathbb{R}$, $Y : \Omega \to \mathbb{R}$ be RVs that are measurable w.r.t $\mathcal{F}$, i.e., $X^{-1}((\infty, x]) \in \mathcal{F}$ and $Y^{-1}((\infty, x]) \in \mathcal{F}$, for any $x \in \mathbb{R}$.

Example. We shall show that $X^2 : \Omega \to \mathbb{R}$ is also a RV

Proof. Fix an $x \in \mathbb{R}$. Consider the set

$$\begin{aligned} B(x) &= \left\{\omega \in \Omega : X^2(\omega) \leq x\right\} \\ &= \left\{\omega \in \Omega : -\sqrt{x} \leq X(\omega) \leq \sqrt{x}\right\} \end{aligned}$$

Let $B_1(x) \triangleq \{\omega \in \Omega : X(\omega) \leq \sqrt{x}\}$ and $B_2(x) \triangleq \{\omega \in \Omega : X(\omega) \leq -\sqrt{x}\}$. Clearly, $B_1(x) \in \mathcal{F}$, and $B_2(x) \in \mathcal{F}$ (**Why?**).
Hence $B(x) = B_1(x) \cap B_2^{\complement}(x) \in \mathcal{F}$, for any $x \in \mathbb{R}$. Therefore, $X^2 : \Omega \in \mathbb{R}$ is also a $\mathcal{F}$-measurable RV. $\square$

Example. We will now show that $X + Y : \Omega \to \mathbb{R}$ is also an $\mathcal{F}$-measurable RV.

Proof. In order to show that $X + Y$ is a RV w.r.t. $(\Omega, \mathcal{F})$, it suffices to show that

$$D(x) \triangleq \{\omega \in \Omega : X(\omega) + Y(\omega) < x\} \in \mathcal{F}, \text{ for any } x \in \mathbb{R}.$$

Now, fix an arbitrary $x \in \mathbb{R}$. Then, the fact that $X(\omega) + Y(\omega) < x$ implies that there exists a rational $q \in \mathbb{Q}$ such that $X(\omega) < q$ and $Y(\omega) < x - q$.
Conversely, if there exists a rational $q \in \mathbb{Q}$ such that $X(\omega) < q$ and $Y(\omega) < x - q$, then this implies that $X(\omega) + Y(\omega) < x$. Hence, we get

$$\{\omega \in \Omega : X(\omega) + Y(\omega) < x\} = \underbrace{\bigcup_{q \in \mathbb{Q}}}_{\text{“there exists”}} \{\omega \in \Omega : X(\omega) < q\} \cap \{\omega \in \Omega : Y(\omega) < x - q\}$$

By the properties of the σ-algebra $\mathcal{F}$, it is immediate that $\{\omega \in \Omega : X(\omega) + Y(\omega) < x\} \in \mathcal{F}$. $\square$

Exercise 3.1. 1. Show that XY is also a RV w.r.t. $\mathcal{F}$.

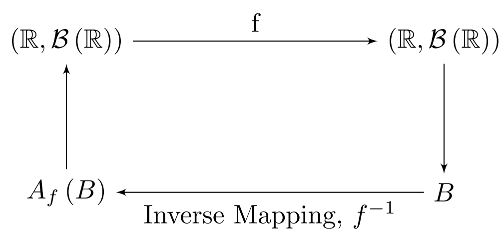
2. Show that $Z \triangleq \min\{X, 0\}$ is a RV w.r.t $\mathcal{F}$.
   Hint: Write the event $\{\omega : Z(\omega) \leq z\}$ as the union of two events, one of which takes on only the values ϕ and Ω.

Let us attempt to come up with a sufficient condition for a function, $f : \mathbb{R} \to \mathbb{R}$ of a random variable $X : \Omega \to \mathbb{R}$ to also be a RV.

Definition (Borel-measurable function)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function. Then, f is said to be *Borel measurable* if

$$A_f(B) \triangleq \{x \in \mathbb{R} : f(x) \in B\} \in \mathcal{B}(\mathbb{R}), \text{ for EVERY Borel set } B \in \mathcal{B}(\mathbb{R}).$$

To depict this pictorially



We had seen a similar picture earlier in the context of RVs. The definition of a Borel-measurable function just replaces $(\Omega, \mathcal{F})$ in the earlier picture with $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

We come to the main result of this section:

Theorem (Functions of RVs)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given probability space, and let $X : \Omega \to \mathbb{R}$ be an $\mathcal{F}$-measurable RV. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a Borel-measurable function, then $f(X) : \Omega \to \mathbb{R}$ is also an $\mathcal{F}$-measurable RV.

Proof. Let $f(X) : \Omega \to \mathbb{R}$ be denoted by the function $g : \Omega \to \mathbb{R}$. We wish to show that for any $B \in \mathcal{B}(\mathbb{R})$, $\left(f^{-1}(X)\right)^{-1}(B) \in \mathcal{F}$.
Now, for a fixed $B \in \mathcal{B}(\mathbb{R})$,

$$\begin{aligned}(f(X))^{-1}(B) &= X^{-1}\left(f^{-1}(B)\right) \\ &= X^{-1}\left(A_f(B)\right) \in \mathcal{F},\end{aligned}$$

since $A_f(B) \in \mathcal{B}(\mathbb{R})$, and from the definition of a RV. □

Remark. Define $\mathcal{L}'$ to be the space of all real-valued RVs on $(\Omega, \mathcal{F}, \mathbb{P})$ which have finite expectation, i.e.,

$$\mathcal{L}' = \{X : X \text{ is an } \mathcal{F}\text{-measurable RV and } \mathbb{E}[X] < \infty\}.$$

Think about why it is true that $\mathcal{L}'$ is a "vector space" over the reals.
Hint:**This follows from the fact that for $X, Y \in \mathcal{L}'$, $\alpha X + \beta Y \in \mathcal{L}'$, for $\alpha, \beta \in \mathbb{R}$ (use the linearity of expectation)**.

3.1 Joint CDFs

Much like we had put down necessary (and sufficient) conditions for a function to be a CDF of a random variable last time, we shall now list down analogous necsssary (and sufficient) conditions for CDFs. We shall restrict our attention to collections of two random variables X_1, X_2. Let us assume that the random vector $\boldsymbol{X} = (X_1, X_2)$ is measurable w.r.t some given probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
Let $F_{X_1,X_2} \simeq F : \mathbb{R}^2 \to [0,1]$ be the joint CDF of $\boldsymbol{X}$. Then, F satisfies

1. $\lim_{x_2 \to \infty} F_{X_1,X_2}(x_1, x_2) = F_{X_1}(x_1)$ and $\lim_{x_1 \to \infty} F_{X_1,X_2}(x_1, x_2) = F_{X_2}(x_2)$
   Proof Sketch: Fix $x_1 \in \mathbb{R}$. Consider the events $B_n(x_1) = \{\omega \in \Omega : X_1(\omega) \leq n\}, n \in \mathbb{N}$. Then, $B_1(x_1) \subseteq B_2(x_2) \subseteq \ldots$. Hence,
   $$\begin{aligned} \lim_{n \to \infty} \mathbb{P}(B_n(x_1)) &= \mathbb{P}\left(\lim_{n \to \infty} B_n(x_1)\right) \\ &= \mathbb{P}\{\omega \in \Omega : X_1(\omega) \leq x_1\} \\ &= F_{X_1}(x_1). \end{aligned}$$
   The proof of the other equation is similar.
2. (Monotonicity) Let $x_1, x_2, x_1', x_2' \in \mathbb{R}$, with $x_1 \leq x_1', x_2 \leq x_2'$. Then $F_{X_1,X_2}(x_1, x_2) \leq F_{X_1,X_2}(x_1', x_2')$.
3. For any $a, b, c, d \in \mathbb{R}$ s.t. $-\infty < a < b < \infty$, $-\infty < c < d < \infty$, we have
   $$\mathbb{P}(a < X_1 \leq b, c < X_2 \leq d) = F(b,d) - F(a,d) - F(b,c) + F(a,c).$$
   Proof: Do it yourself! Draw a picture of the region of interest in $\mathbb{R}^2$, and avoid double counting.
4. F is continuous from above and from the right, i.e.,
   $$\lim_{h \to 0^+, k \to 0^+} F(x_1 + h, x_2 + k) = F(x_1, x_2),\ x_1, x_2 \in \mathbb{R}.$$
   Proof Sketch: Consider sets $A_{mn}(x_1, x_2) \triangleq \left\{\omega \in \Omega : X_1(\omega) \leq x_1 + \frac{1}{m}, X_2(\omega) \leq x_2 + \frac{1}{n}\right\}$. Split up the limit $\lim_{m \to \infty, n \to \infty} F\left(x_1 + \frac{1}{m}, x_2 + \frac{1}{n}\right)$ into iterated limits, and use the definition of $A_{m,n}(x_1, x_2)$.

Exercise 3.2. Show that $F : \mathbb{R}^2 \to [0,1]$ defined by
$$F(x,y) = \begin{cases} 0,\ x < 0, \\ (1 - e^{-x})\left(1/2 + \frac{1}{\pi}\tan^{-1} y\right),\ \text{if } x \geq 0 \end{cases}$$
is a valid joint CDF.

3.2 Transformation of Random Vectors

3.2.1 Simulating CDFs

In this section, we are interested in transforming the Unif([0, 1]) random variable into any other random variable, whose CDF is known. This is of use in settings where we wish to *sample* from a new random variable whose distribution is known.

In class, we have prooved the following lemma:

Lemma 3.3. Let $U \sim \text{Unif}([0,1])$. Let $F : \mathbb{R} \to [0,1]$ be the CDF of the random variable we wish to simulate (or sample from). We will assume, initially, that $F$is continuous and strictly increasing. Consider the RV $X = F^{-1}(U)$. Then X has the distribution F.

It turns out that we can define $F^{-1} : [0,1] \to \mathbb{R}$ for a general $F(\cdot)$ as

$$F^{-1}(x) = \inf\{u : F(u) \geq x\}, \text{ for } x \in [0,1].$$

Exercise 3.4. Propose a transformation to transform the Unif([0, 1]) RV to the Ber(p) RV.

Corollary. Let X be a given RV with distribution $F : \mathbb{R} \to [0,1]$. Consider the RV $Y = F(X)$. Then Y has the same distribution as the Unif([0, 1]) RV.

Proof. Reverse the proof of the previous lemma. □

Exercise 3.5. Let $U \sim \text{Unif}([0,1])$. Show that $Y = -\frac{1}{\lambda}\ln U$ is exponentially distributed with parameter λ. Compare Y with a $\tilde{Y}$ generated by the lemma above. What do you conclude?

3.2.2 Transformation of Random Vectors

Let us recall the theorem seen in class.

Theorem. Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ have joint density f. Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable and injective. Then $Y = g(X)$ has the density

$$f_Y(y) = \begin{cases} f_X\left(g^{-1}(y)\right)\left|\det\left(J_{g^{-1}}(y)\right)\right|, & \text{if } y \text{ is in the range of } g, \\ 0, & \text{o.w.} \end{cases}$$

Example. Let (X, Y) have joint density $f : \mathbb{R}\setminus\{0\} \times \mathbb{R}\setminus\{0\} \to [0, \infty)$. Find the density of $Z = XY$.
Solution: Let $g(x, y) = (xy, x) = (u, v)$. Now,

$$g^{-1}(u,v) = \left(v, \frac{u}{v}\right) \qquad \text{[Note that } g^{-1}(\cdot) \text{ exists]},$$

$$J_{g^{-1}}(u,v) = \begin{pmatrix} 0 & 1 \\ \frac{1}{v} & -\frac{u}{v^2} \end{pmatrix}, \text{ with } \det\left(J_{g^{-1}}(u,v)\right) = -\frac{1}{v}.$$

Therefore, $f_{U,V}(u,v) = f_{X,Y}\left(v, \frac{u}{v}\right) \cdot \frac{1}{v}$ for $u, v \in \mathbb{R}\setminus\{0\}$.

3.3 Supplementary Exercises

Assume a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Exercise 3.6. Suppose U, V are jointly uniformly distributed RVs over the square with corners $(0,1), (1,0), (1,1)$, and $(0, 1)$, i.e.,

$$f_{U,V}(u,v) = \begin{cases} 1, & \text{if } 0 \leq u \leq 1,\ 0 \leq v \leq 1, \\ 0, & \text{o.w.} \end{cases}$$

1. Write down the marginals $f_U(u)$ and $f_V(v)$ for all $u, v \in \mathbb{R}$. Are U, V independent?

2. Define a new RV $X = UV$. Find the pdf of X.
   Hint:**Draw a picture of the region $\{UV \leq x\}$ for some $x > 0$. Can you compute the area of this region (do not write down the integral blindly!)? Differentiate this are w.r.t. x to get the pdf.**

Exercise 3.7. Let X and Y have a joint pdf given by

$$f(x, y) = x + y,\ 0 \leq 1,\ 0 \leq y \leq 1.$$

Are X and Y independent?

Exercise 3.8. Let $U \sim \text{Unif}([0, 1])$. What is the pmf of $\lfloor nU \rfloor + 1$, where $n \in \mathbb{N}$?
Note: For an $x \in \mathbb{R}$, $\lfloor x \rfloor$ is the largest integer less than or equal to x.