E2:202 Random Processes	**Nov. 20, 2020**

Tutorial 7: Convergence of Random Variables

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In this tutorial, we will go over the definitions of the different kinds of convergence of random variables, the proofs of some implications, and a some examples.

All through, we assume that we have a defined probability space, $(\Omega, \mathcal{F}, \mathbb{P})$.

The first kind of convergence that we are interested in dealing with is called "convergence everywhere" — a very strong form of pointwise convergence, that is most often too much to ask for.

Definition (Convergence Everywhere)**.** A sequence of $\mathcal{F}$-measurable random variables $(X_n : n \in \mathbb{N})$ is said to converge to an $\mathcal{F}$-measurable random variable X everywhere (or converge point-wise to X) if

$$\lim_{n\to\infty} X_n(\omega) = X(\omega) \quad \forall \omega \in \Omega.$$

Recall that a sequence of real numbers $(a_n : n \in \mathbb{N})$ converges to the real number $a \in \mathbb{R}$ (or $\lim_{n\to\infty} a_n = a$) if

for every $\epsilon > 0$, there exists an $N(\epsilon)$, such that for all $n \geq N(\epsilon)\, |a_n - a| \leq \epsilon$.
(or equivalently)
for every $k \in \mathbb{N}$, there exits an $N(k)$, such that for all $n \geq N(k)$, $|a_n - a| \leq \frac{1}{k}$.

We must read the convergence of $(X_n : n \in \mathbb{N})$ as the convergence of the real sequence $X_n(\omega)$ to the real number $X(\omega)$ for each *fixed* $\omega \in \Omega$.

Example.

1. Let $(X_n : n \in \mathbb{N})$ be a sequence of random variables on the standard unit interval probability space $(\Omega = [0,1], \mathcal{F} = \mathcal{B}([0,1], \mathbb{P}((a,b)) = b - a),$ for $0 \leq a \leq b \leq 1)$, defined by $X_n(\omega) = \omega^n$.

   This sequence converges for all $\omega \in \Omega$ to the limit random variable

$$X(\omega) = \begin{cases} 0, & \omega \neq 1, \\ 1, & \omega = 1. \end{cases}$$

2. Very that the same holds for $X_n(\omega) = n\omega^n$ on the same probability space.

Exercise 7.1 (Short Exercises on the limits of real sequences)**.**

1. Show (using the definition of a limit given earlier) that if sequences $(x_n \in \mathbb{R} : n \in \mathbb{N})$, $(y_n \in \mathbb{R} : n \in \mathbb{R})$, are such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then

$$\lim_{n\to\infty} x_n y_n = xy.$$

2. Show that $\lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{x}$ if, in addition, $x_n \neq 0$, for any $n \in \mathbb{N}$ and $x \neq 0$.

3. (Cesàro's mean): Show that $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n = x$.

4. (Partial Sums): Consider a sequence of non-negative real numbers $(a_n : n \in \mathbb{N})$ such that

$$\sum_{n=1}^{\infty} a_n \triangleq \lim_{N\to\infty} \sum_{n=1}^{N} a_n = M < \infty.$$

Prove that

(a) $\lim\limits_{N\to\infty} \sum_{n=N}^{\infty} a_n = 0$, and

(b) $\lim\limits_{n\to\infty} a_n = 0$.

These results show that for a "summable" non-negative sequence $(a_n : n \in \mathbb{N})$, $\lim\limits_{n\to\infty} a_n = 0$ and that the "tail sums" $(\sum_{n=N}^{\infty} a_n)$ also go to zero.

We now define almost sure convergence.

Definition (Almost Sure Convergence)**.** A sequence of $\mathcal{F}$-measurable random variables $(X_n : n \in \mathbb{N})$ is said to converge a.s. (almost surely) to an $\mathcal{F}$-measurable random variable X, if

$$\mathbb{P}\left(\left\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\}\right) = 1.$$

Remark. Note that this is different from convergence everywhere. Here, we require the limit to hold on a set of probability one.

Recall the example seen in class: For the standard unit-interval probability space, we define $X_n(\omega) = 1_{\left\{\omega \in \left[0,\frac{1}{n}\right)\right\}}$ for $n \in \mathbb{N}$. This sequence clearly does not converge everywhere to $X(\omega) = 0,\ \forall \omega \in \Omega$ because $X_n(0) = 1$ for all $n \in \mathbb{N}$. However, we would like to show that

$$X_n \xrightarrow[n\to\infty]{\text{a.s}} X = 0 \quad \text{(note this notation)}$$

We will now introduce a formal way of looking at almost sure convergence. We note that

$$\left\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\} = \left\{\omega : \text{for every } k \in \mathbb{N}, \text{ there exists an } N(k) \text{ such that } \forall n \geq N(k), |X_n(\omega) - X(\omega)| \leq \frac{1}{k}\right\}$$
$$\triangleq A^X$$

Let us translate what we have written in words to the language of mathematics:

$$A^X = \underbrace{\bigcap_{k\in\mathbb{N}}}_{\text{for all } k} \underbrace{\bigcup_{N(k)\in\mathbb{N}}}_{\text{there exists } N(k)} \underbrace{\bigcap_{n\geq N(k)}}_{\text{for all } n \geq N(k)} \left\{\omega \in \Omega : |X_n(\omega) - X(\omega)| \leq \frac{1}{k}\right\}.$$

In particular, for a fixed $k \in \mathbb{N}$, we define

$$A_k^X \triangleq \bigcup_{N(k)\in\mathbb{N}} \bigcap_{n\geq N(k)} \left\{\omega \in \Omega : |X_n(omega) - X(\omega)| \leq \frac{1}{k}\right\}.$$

Further, if we define

$$A_{n,k}^X \triangleq \left\{\omega \in \Omega : |X_n(\omega) - X(\omega)| \leq \frac{1}{k}\right\},$$

we note that A_k^X is the event $\left\{A_{n,k}^X \text{ infinitely often}\right\}$! Hence, for a <u>fixed</u> $k \in \mathbb{N}$, we would like the event

$$\left\{\omega \in \Omega : |X_n(\omega) - X(\omega)| \leq \frac{1}{k}\right\}$$

to occur infinitely often w.p. 1. If this holds for every $k \in \mathbb{N}$, we say that $X_n \xrightarrow[n\to\infty]{\text{a.s}} X$.

With this picture in mind, let us prove the almost sure convergence of the sequence that we saw earlier, to the random variable that is identically zero.

Now, fix a $k \in \mathbb{N}$. The set

$$\begin{aligned}
A_k^X &= \bigcup_{N(k)=1}^{\infty} \bigcap_{n=N(k)}^{\infty} \left\{\omega \in \Omega : |X_n(\omega)| \leq \frac{1}{k}\right\} \\
&= \bigcup_{N(k)=1}^{\infty} \bigcap_{n=N(k)}^{\infty} \{\omega : |X_n(\omega)| = 0\} \\
&= \bigcup_{N(k)=1}^{\infty} \bigcap_{n=N(k)}^{\infty} \left[\frac{1}{n}, 1\right] \\
&= [0, 1).
\end{aligned}$$

Hence, $\mathbb{P}\left(A_k^X\right) = 1$ for <u>any</u> $k \in \mathbb{N}$. Hence $X_n \xrightarrow[n\to\infty]{\text{a.s.}} 0$.

Example. For an example of a sequence of random variables that does not converge a.s. to any random variable, consider the following example on the standard unit-interval probability space: (courtesy Prof. Bruce Hajek's book)

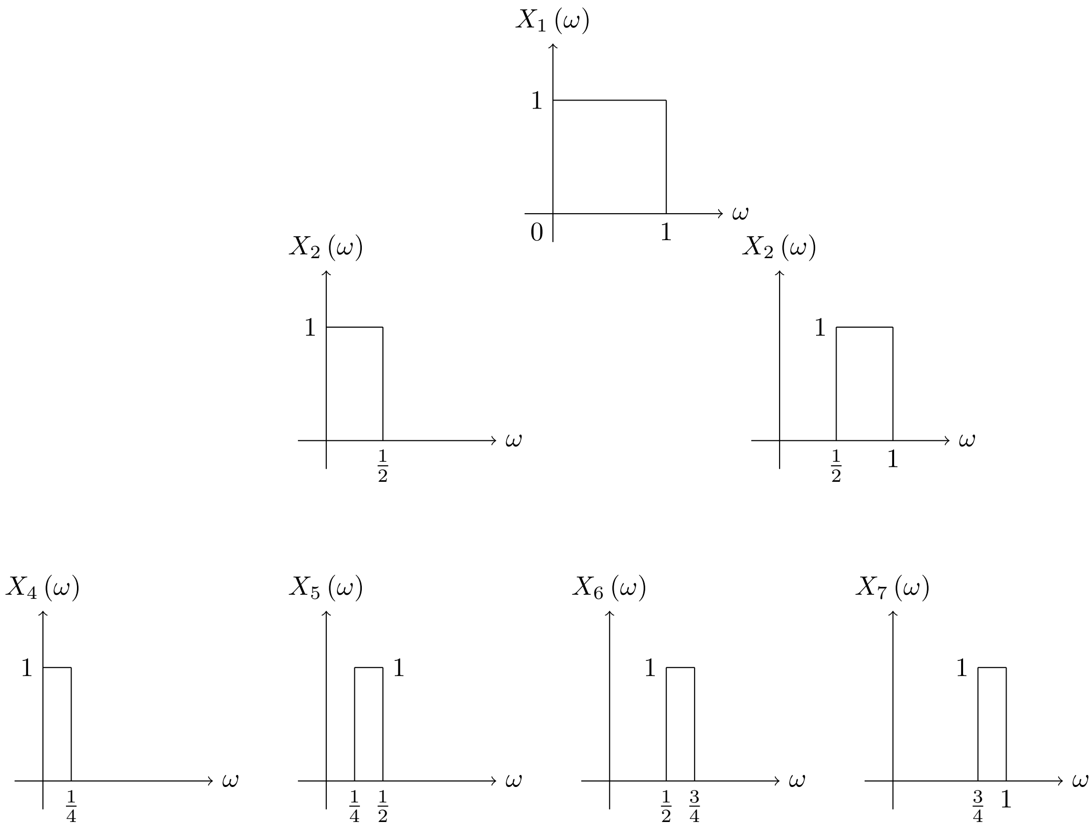

Figure 7.1: Moving rectangles (courtesy Prof. Bruce Hajek's book)

Remark. As an exercise in writing, write down analytical expressions for these random variables as a function of ω as succintly as possible.

Here, note that for any $\omega \in [0,1]$, there exist infinitely many X_ns such that $X_n(\omega) = 0$, and infinitely many (other) X_ns such that $X_n(\omega) = 1$. Hence, X_n does not converge almost surely to any random variable.

We now move ahead to define our third form of convergence – convergence in probability.

Definition (Convergence in Probability)**.** A sequence of $\mathcal{F}$-measurable random variables $(X_n : n \in \mathbb{N})$ is said to converge in probability to the $\mathcal{F}$-measurable random variable X, if for *any* $\epsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}\left(\{\omega \in \Omega : |X_n(\omega) - X(\omega)| > \epsilon\}\right) = 0.$$

Remark. A simple illustration of convergence in probability is the moving rectangles example we saw earlier, where the random variables now converge in probability (not a.s.) to the identically zero random variable.

We now wish to show that if $X_n \xrightarrow[n\to\infty]{\text{a.s}} X$, then $X_n \xrightarrow[n\to\infty]{p} X$.

Proof. We are given that $X_n \xrightarrow[n\to\infty]{\text{a.s}} X$. This implies that for any $\epsilon > 0$,

$$\begin{aligned}
\mathbb{P}\left(A^X_{n,\epsilon} \text{ i.o}\right) &= 1 \quad (A^X_{n,\epsilon} \text{ analogous to } A^X_{n,k} \text{ defined earlier}) \\
\Rightarrow \mathbb{P}\left(\cup_{N(\epsilon)=1}^{\infty} \cap_{n\geq N(\epsilon)} \{\omega : |X_n(\omega) X(\omega)| \leq \epsilon\}\right) &= 1 \\
\Rightarrow \lim_{N\to\infty} \mathbb{P}\left(\cap_{n\geq N} \{\omega : |X_n(\omega) X(\omega)| \leq \epsilon\}\right) &= 1 \\
\Rightarrow \lim_{N\to\infty} \mathbb{P}\left(\{\omega : |X_n(\omega) X(\omega)| \leq \epsilon\}\right) &= 1 \quad \text{(why?)}
\end{aligned}$$

Hence, $X_n \xrightarrow[n\to\infty]{p} X$. ☐

Remark. The converse does not necessarily hold. That is, $X_n \xrightarrow[n\to\infty]{p}$ does not imply $X_n \xrightarrow[n\to\infty]{\text{a.s}} X$ in general.
Recall the example of the "moving rectangles".

We now move to the fourth kind of convergence that we are interested in.

Definition ($\mathcal{L}^p$ convergence)**.** A sequence of $\mathcal{F}$-measurable random variables $(X_n : n \in \mathbb{N})$ is said to converge in $\mathcal{L}^p$ to the $\mathcal{F}$-measurable random variable X if for some $p \geq 1$,

$$\lim_{n\to\infty} \mathbb{E}\left[|X_n - X|^p\right] = 0.$$

We will be primarily interested in the cases where $p = 1$ or $p = 2$.

Example (Anchored Rectangles)**.** Consider random variables $(X_n : n \in \mathbb{N})$ defined on the standard unit-interval probability space, where

$$X_n(\omega) = n1_{\left\{\omega\in\left[0,\frac{1}{n}\right)\right\}}.$$

Show that $X_n \xrightarrow[n\to\infty]{\text{a.s.}} 0$ but $X_n \not\xrightarrow[n\to\infty]{\mathcal{L}^p} 0$ for any $p \geq 1$.

Remark.
1. A simple application of Markov's inequality leads to the lemma that convergence in $\mathcal{L}^p$ implies convergence in probability.
2. It is also easy to note that in the "moving rectangles" example,

$$X_n \xrightarrow[n\to\infty]{\mathcal{L}^p} 0.$$

Hence, the (partial) picture of the relationship between the different kinds of convergences is as follows:

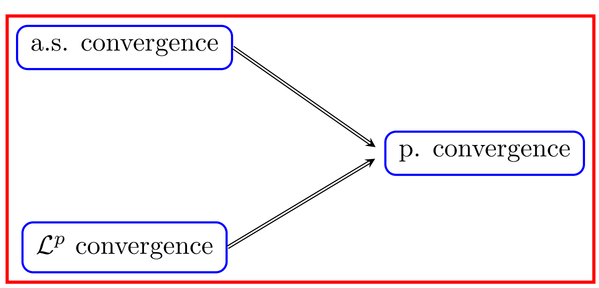

Figure 7.2: Partial Picture of relationship between convergences