E2:202 Random Processes **Nov. 21, 2020**

Tutorial 8: Problems on convergence of RVs and BC Lemma

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We shall now see a few example problems on the almost sure, $\mathcal{L}^2$ and "in distribution" convergence of random variables.

Example 8.1. Let $U \sim \text{Unif}([0,1])$ and let $X_n = \frac{(-1)^n U}{n}$ for $n \geq 1$. Let the probability space be the standard unit-interval probability space

1. Show that $(X_n : n \in \mathbb{N})$ converges almost surely.

   Solution. Fix an $\epsilon > 0$. Our claim is that $X_n \xrightarrow[n\to\infty]{a.s.} 0$.

   We wish to show that the set

   $$A_\epsilon^0 \triangleq \bigcup_{N(\epsilon)\geq 1} \bigcap_{n \geq N(\epsilon)} \{\omega : |X_n(\omega)| \leq \epsilon\}$$

   is of probability 1. To see this, note that

   $$\begin{aligned} A_\epsilon^0 &= \bigcup_{N(\epsilon)\geq 1} \bigcap_{n \geq N(\epsilon)} \{\omega : U(\omega) \leq \epsilon\} \\ &= \bigcup_{N(\epsilon)\geq 1} \bigcap_{n \geq N(\epsilon)} [0, \min\{n\epsilon, 1\}] \\ &= [0,1] \quad \text{(Fill in the details)} \end{aligned}$$

   Hence, $\mathbb{P}\left(A_\epsilon^0\right) = 1,\ \forall \epsilon > 0$, showing that $X_n \xrightarrow[n\to\infty]{\text{a.s.}} 0$. □

2. Show that the sequence converges in the mean squared sense.

   Solution. Observe that

   $$\begin{aligned} \mathbb{E}\left[|X_n^2 - 0|^2\right] &= \mathbb{E}\left[|X_n|^2\right] \\ &= \frac{1}{n^2}\mathbb{E}\left[U^2\right] \\ &= \frac{1}{3n^2}, \text{ and hence,} \\ \lim_{n\to\infty} \mathbb{E}\left[|X_n - 0|^2\right] &= 0. \end{aligned}$$

   Thus $X_n \xrightarrow[n\to\infty]{\text{m.s.}} 0$. □

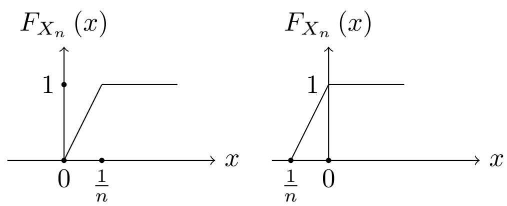
Recall that the definition of convergence in distribution, given $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition. A sequence $(X_n : n \in \mathbb{N})$ of $\mathcal{F}$-measurable random variables is said to converge in distribution to an $\mathcal{F}$-measurable random variable X if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x),$$

for <u>every</u> point of continuity, $x \in \mathbb{R}$ of $F_X(\cdot)$.

With this definition in mind, can you show formally that in the example above, $X_n \xrightarrow[n\to\infty]{\text{d.}} 0$?



(n even) (n odd)

One can see that

$$\begin{aligned} F_{X_n}(x) &\xrightarrow[n\to\infty]{0}, \ \forall x < 0, \text{ and} \\ F_{X_n}(x) &\xrightarrow{n\to\infty} 1, \ \forall x \geq 0. \end{aligned}$$

Hence, $X_n \xrightarrow[n\to\infty]{\text{d.}} 0$ as 0 is the only point of discontinuity of $F_X(\cdot)$, where $X \equiv 0$.

Example 8.2 (Courtesy Karthik P. N.)**.** Let $W_1, W_2, \ldots$ be a sequence of i.i.d. $\mathcal{N}\left(0, \sigma^2\right)$ random variables. Let X_0 and define

$$X_{n+1} = \frac{X_n + W_{n+1}}{2} \ n \geq 0$$

Which random variable does X_n converge in distribution to?

Solution. By iterating (or rolling-out) the equation for X_{n+1}, we obtain that

$$X_n = \frac{W_1}{2^n} + \frac{W_2}{2^{n-1}} + \ldots + \frac{W_n}{2}.$$

Since $(W_i : i \in \mathbb{N})$ are i.i.d, $X_n \sim \mathcal{N}\left(0, \sigma^2\left(\sum\limits_{i=1}^{n} \frac{1}{4^i}\right)\right)$. Therefore,

$$\begin{aligned} \mathbb{P}\left(X_n \leq x\right) &= \mathbb{P}\left(\frac{X_n - 0}{\sqrt{\sigma^2 \sum_{i=1}^n \frac{1}{4^i}}} \leq \frac{x}{\sqrt{\sigma^2 \sum_{i=1}^n \frac{1}{4^i}}}\right) \\ &= \phi\left(\frac{x}{\sqrt{\sigma^2 \sum_{i=1}^n \frac{1}{4^i}}}\right) \ \forall x \in \mathbb{R}. \end{aligned}$$

Hence,

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}\left(X_n \le x\right) &= \lim_{n\to\infty} F_{X_n}\left(x\right) \\ &= \phi\left(\lim_{n\to\infty} \frac{x}{\sqrt{\sigma^2 \sum_{i=1}^{n} \frac{1}{4^i}}}\right) \quad \text{(since } \phi\left(\cdot\right) \text{ is continuous)} \\ &= \phi\left(\frac{\sqrt{3}x}{\sigma}\right), \ \forall x \in \mathbb{R}.\end{aligned}$$

Hence, $X_n \xrightarrow[n\to\infty]{\text{d.}} \mathcal{N}\left(0, \sigma^2/3\right)$. □

So far, in the context of almost sure convergence, we have seen examples where the probability space was explicitly defined. This allowed us to use the "first-principles" definition of almost sure convergence to solve the problems.

Our goal now is to demonstrate how convergence in probability can be used to prove almost sure convergence in select case, by using the powerful Borel-Cantelli lemmas.

Lemma (Borel-Cantelli)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given probability space.

1. For a sequence of events, $(A_n : n \in \mathbb{N})$, if $\sum\limits_{n\in\mathbb{N}} \mathbb{P}\left(A_n\right) < \infty$, then

$$\mathbb{P}\left(A_n \text{ i.o.}\right) = 0.$$

2. For a sequence of mutually independent events, $(A_n : n \in \mathbb{N})$, if $\sum\limits_{n\in\mathbb{N}} \mathbb{P}\left(A_n\right) = \infty$, then

$$\mathbb{P}\left(A_n \text{ i.o.}\right) = 1.$$

Proof. 1. We know that

$$\begin{aligned}\mathbb{P}\left(A_n \text{ i.o.}\right) &= \mathbb{P}\left(\bigcap_{N=1}^{\infty} \bigcup_{n\ge N} A_n\right) \\ &= \lim_{N\to\infty} \mathbb{P}\left(\bigcup_{n\ge N} A_n\right) \quad \text{(Why?)} \\ &\le \lim_{N\to\infty} \sum_{n=N}^{\infty} \mathbb{P}\left(A_n\right) \quad \text{(Union bound)} \\ &= 0,\end{aligned}$$

where the last equality is true since we are given that $\sum_{n\in\mathbb{N}} \mathbb{P}\left(A_n\right) < \infty$ and hence the tail sums go to zero.

2. We want to show that $\mathbb{P}(A_n \text{ i.o.}) = 1$. Equivalently, we show that $\mathbb{P}\left(\{A_n \text{ i.o.}\}^{\mathsf{C}}\right) = 0$.

$$\begin{aligned}
\mathbb{P}\left(\{A_n \text{ i.o.}\}^{\mathsf{C}}\right) &= \mathbb{P}\left(\bigcup_{N=1}^{\infty} \bigcap_{n \geq N} A_n^{\mathsf{C}}\right) \\
&= \lim_{N\to\infty} \mathbb{P}\left(\bigcap_{n\geq N} A_n^{\mathsf{C}}\right) \quad \textbf{(Why?)} \\
&= \lim_{N\to\infty} \prod_{n=N}^{\infty} \left(1 - \mathbb{P}(A_n)\right) \quad (A_n\text{s are independent}) \\
&\leq \lim_{N\to\infty} \prod_{n=N}^{\infty} e^{-\mathbb{P}(A_n)} \quad \text{since } 1 - x \leq e^{-x}\ \forall x \in \mathbb{R} \\
&= e^{-\lim\limits_{N\to\infty} \sum_{n=N}^{\infty} \mathbb{P}(A_n)} \\
&= 0
\end{aligned}$$

since $\sum_{n\in\mathbb{N}} \mathbb{P}(A_n) = \infty$ by the hypothesis. Hence,

$$\mathbb{P}(A_n \text{ i.o.}) = 1.$$

$\square$

Exercise 8.3. Suppose that $((X_1, Y_1), (X_2, Y_2), \ldots)$ is a sequence of random vectors such that $\mathbb{P}(\{X_k \geq Y_k\}) = \alpha^k$ for some $0 < \alpha < 1$. Show that $\mathbb{P}(\{X_k \geq Y_k\} \text{ i.o.}) = 0$.

How do we use the Borel-Cantelli lemmas to prove almost sure convergence from the knowledge of convergence in probability? The recipe is as follows.

1. For every fixed $\epsilon > 0$, obtain an expression for, or an upper bound on, the probability $\mathbb{P}(|X_n - X| \geq \epsilon)$, where X is the (guessed) limit random variable.

2. If $\sum\limits_{n=1}^{\infty} \mathbb{P}(|X_n - X| \geq \epsilon) < \infty$, then by the Borel-Cantelli Lemma (1), we can say that $\mathbb{P}(\{|X_n - X| \geq \epsilon\} \text{ i.o.}) = 0$.
   If this holds for every $\epsilon > 0$, then this immediately implies that $X_n \xrightarrow[n\to\infty]{\text{a.s.}} X$.

Use the first principles method of proving almost sure convergence to ascertain that the last statement holds.

If on the other hand, the X_ns are independent, and we are able to show that $\sum_{n=1}^{\infty} \mathbb{P}(|X_n - X| \geq \epsilon) = \infty$, then

$$\mathbb{P}(\{|X_n - X| \geq \epsilon\} \text{ i.o.}) = 1$$

for some $\epsilon > 0$. It then follows that $X_n \not\xrightarrow[n\to\infty]{\text{a.s.}} X$.

Let us now look at an example to see this procedure in action.

Example 8.4. $(X_n : n \in \mathbb{N})$ is a sequence of independent random variables with marginal pmfs given by

$$\mathbb{P}\left(X_n = \frac{1}{2}\left(1 - \frac{1}{n}\right)\right) = \mathbb{P}\left(X_n = \frac{1}{2}\left(1 + \frac{1}{n}\right)\right) = \frac{1}{2}.$$

1. Show that the sequence converges almost surely.

   Solution. First, we note that for a fixed $\epsilon > 0$,

$$\mathbb{P}\left(\left\{\omega : |X_n(\omega) - \frac{1}{2}| \leq \epsilon\right\}\right) = \mathbb{P}\left(\frac{1}{2} - \epsilon \leq X_n \leq \frac{1}{2} + \epsilon\right)$$
$$= \begin{cases} 0, & \text{if } \epsilon < \frac{1}{2n} \ \left(\text{ or } n < \frac{1}{2\epsilon}\right) \\ 1, & \text{o.w.} \end{cases}$$

   Hence, it is immediate that

$$\lim_{n\to\infty} \mathbb{P}\left(\left\{\omega : |X_n(\omega) - \frac{1}{2}| > \epsilon\right\}\right) = 0.$$
$$\Rightarrow X_n \xrightarrow[n\to\infty]{\text{p.}} \frac{1}{2}.$$

   Now, observe that

$$\sum_{n=1}^{\infty} \mathbb{P}\left(|X_n - \frac{1}{2}| > \epsilon\right) = \frac{1}{2\epsilon} < \infty$$

   Hence, $X_n \xrightarrow[n\to\infty]{\text{a.s.}} \frac{1}{2}$. □

2. Check if $(X_n : n \in \mathbb{N})$ converges in $\mathcal{L}^2$ (mean-squared convergence)

   Solution. We have that

$$\mathbb{E}\left[\left(X_n - \frac{1}{2}\right)^2\right] = \frac{1}{2}\frac{1}{4n^2} + \frac{1}{2}\frac{1}{4n^2}$$
$$= \frac{1}{4n^2} \xrightarrow[n\to\infty]{0} .$$

   Hence, $X_n \xrightarrow[n\to\infty]{\text{m.s.}} 0$. □

Remark. In the example above, the almost sure convergence implies convergence in probability and convergence in distribution. However, it <u>does not</u> imply (nor is implied by) convergence in $\mathcal{L}^2$. The second property, hence, needs to be checked independently.