E2:202 Random Processes **Dec. 04, 2020**

Tutorial 10: An introductory look at Random Processes

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In this tutorial, we shall discuss the definition of a random process and review some of the quantities (means, correlation, etc.) associated with a random process. We will also briefly discuss random walks.

Definition. A random process X is an indexed collection $X = (X_t : t \in \mathrm{T})$ of random variables, all on the same probability space, $(\Omega, \mathcal{F}, P)$.

Remark.

1. If $\mathrm{T} = \mathbb{Z}$, then X is called a discrete-time random process.
2. If $\mathrm{T} = \mathbb{R}$, or if T is an interval of $\mathbb{R}$, then X is called a continuous-time random process.
3. Often, it is useful to view a random process as $X : \Omega \to \mathfrak{X}^{\mathrm{T}}$. For each $\omega \in \Omega$, $X_t(\omega)$ is a function of t, called the sample path corresponding to ω.

Associated with a random process X are quantities $\mu_X(t)$, $R_X(s,t)$ and $C_X(s,t)$ for $s,t \in \mathrm{T}$. Refer to the lecture notes for the definitions of these quantities.

Example. Let $U = (U_k : k \in \mathbb{Z})$ be a random process such that the U_ks are independent and $\mathbb{P}(U_k = 1) = \mathbb{P}(U_k = -1) = \frac{1}{2}$. Note that

$$R_U(k,l) = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{o.w.} \end{cases} \quad (\text{for } k,l \in \mathbb{Z})$$

$$R_X(s,t) = \begin{cases} 1, & \text{if } \lfloor s \rfloor = \lfloor t \rfloor \\ 0, & \text{o.w.} \end{cases} \quad (\text{for } s,t \in \mathbb{R}).$$

Exercise 10.1. Let A, B be independent $\mathcal{N}(0,1)$ random variables. Suppose that $X_t = A + t + Bt^2$. Can you write down the pdf of X_t for $t \in \mathbb{R}$?

10.1 A brief look at random walks

A random walk is a discrete-time random process (or stochastic process) $X = (X_n : n \in \mathbb{N} \cup \{0\})$ with the initial condition $X_0 = 0$, and the update rule

$$X_{n+1} = X_n + U_n, \quad n \in \mathbb{N} \cup \{0\},$$

where $U = (U_n : n \in \mathbb{N} \cup \{0\})$ is some i.i.d. process.

Remark. Note that the *evolution* of the process is of the form

$$X_{n+1} = f(X_n, U_n) \text{ for } (U_n : n \in \mathbb{N})$$

being an i.i.d. process. We shall return to this observation, later, when we speak about Markov Chains.

The random walk has the "independent increment property". To see this, note that for $s, t \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} X_t &= X_t - X_0 \\ &= (X_t - X_s) + (X_s - X_0). \\ \text{Hence, } X_t - X_s &= (X_t - X_0) - (X_s - X_0) \\ &= \sum_{r=s}^{t-1} U_r \end{aligned}$$

which is a function of $U_s, \ldots, U_{t-1}$, while $X_s - X_0$ is a function of $U_0, \ldots, U_{s-1}$. Since U is an i.i.d. process, the independent increment property holds.

Also, note that if $\mathbb{E}[[]\, U_1] = \mu$ and $\text{var}(U_1) = \sigma^2$,

$$\begin{aligned} \mu_X(t) &= \mathbb{E}[X_t] = t\mu \\ R_X(t,t)\text{var}(X_t) &= t\sigma^2, \\ \text{and } R_X(t,s) &= \mathbb{E}[X_t - X_s] \quad \text{(assume WLOG that } t > s) \\ &= \mathbb{E}[(X_t - X_0)(X_s - X_0)] \\ &= \mathbb{E}[(X_s - X_0)(X_t - X_s + X_s - X_0)] \\ &= \mathbb{E}[(X_s - X_0)(X_t - X_s)] + \mathbb{E}\left[(X_s - X_0)^2\right] \\ &= \mathbb{E}[X_s - X_0]\,\mathbb{E}[X_t - X_s] + \mathbb{E}\left[(X_s - X_0)^2\right] \quad \textbf{(Why?)} \\ &= st\mu^2 + s\sigma^2. \end{aligned}$$

Remark. It is immediate that if $\mu \neq 0$ or $\sigma \neq 0$, then the random walk above is not stationary.

Now, we will take a look at an interesting (and popular) problem in simple random walks on the integers.

Definition. A random walk is "simple" if $\mathbb{P}(U_i = 1) = p = 1 - \mathbb{P}(U_i = 0), \forall i \in \mathbb{N} \cup \{0\}$.

10.1.1 The Gambler's Ruin Problem

Let us consider the evolution of a gambler's wealth as a simple random walk on the integers.

Let X_n be the number of units of wealth a gambler has at time n and let $X_0 = k \geq 0$. The gambler wishes to accumulate b units of wealth for $b \geq k$, before it reaches a wealth of 0 units. Let us call this event a "success".
We wish to compute the probability $\mathbb{P}(\textit{success} \mid X_0 = k) \triangleq s_k$.

To this end, note that

$$\begin{aligned} s_k &= \mathbb{P}(U_1 = 1)\,\mathbb{P}(\textit{success} \mid U_1 = 1, X_0 = k) + \mathbb{P}(U1 = -1)\,\mathbb{P}(\textit{success} \mid U_1 = -1, X_0 = k) \\ \text{i.e., } s_k &= p.s_{k+1} + (1-p)\,s_{k-1} \text{ with } s_1 = 1 \text{ and } s_0 = 0. \end{aligned}$$

For $p \neq 1/2$, this recurrence relation can be explicitly solved to yield

$$s_k = \frac{1 - \left(\frac{1-p}{p}\right)^k}{1 - \left(\frac{1-p}{p}\right)^b}, \quad 0 \leq k \leq b.$$

Further, when $p = 1/2$, the solution to the recurrence relation is

$$s_k = \frac{k}{b}, \quad 0 \leq k \leq b.$$

Suppose that $p > 1/2$, note that

$$\lim_{b \to \infty} s_k = 1 - \left(\frac{1-p}{p} \right)^k,$$

which decreases geometrically with the initial wealth k.