E2:202 Random Processes **Dec. 11, 2020**

Tutorial 11: A first look at DTMCs

Lecturer: Parimal Parag *TA: Arvind* *Scribes: Krishna Chaythanya KV*

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In all the tutorials that follow, we will primarily be concerned with DTMCs with finite and (sometimes) countable state spaces.

Definition (DTMC)**.** A discrete time random process $X = (X_n : n \in \mathbb{N} \cup \{0\})$ is called a DTMC if it holds that for any $n \geq 1$, and for all $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$,

$$P\left(X_n = x_n \mid X_{n-1} = x_{n-1}, \ldots, X_0 = x_0\right) = P\left(X_n = x_n \mid X_{n-1} = x_{n-1}\right).$$

Remark.

1. We denote the transition probability $P\left(X_n = y \mid X_{n-1} = x\right)$ as $P_{xy}(n)$.
2. However, our interest will be in time-homogenous DTMCs for which $P_{xy}(n) = P_{xy}, \forall n \geq 1$.

Example 11.1. Herein, we show that if $X : \Omega \to \mathcal{X}^{\mathbb{N}}$ is a time-homogeneous DTMC, then the invariance of the distribution of X_n with n is sufficient for X to be stationary.

Solution. We need to show that for any collection $n_1, n_2, \ldots, n_N$ of indices,

$$P\left(X_{n_1} = i_1, X_{n_2} = i_2, \ldots, X_{n_N} = i_N\right) = P\left(X_{n_1+t} = i_1, X_{n_2+t} = i_2, \ldots, X_{n_N+t} = i_N\right) \ \forall t \in \mathbb{N}.$$

To see this, note that

$$\begin{aligned} P\left(X_{n_1} = i_1, \ldots, X_{n_N} = i_N\right) &= P\left(X_{n_1} = i_1\right) P_{i_1 i_2}^{(n_2-n_1)} P_{i_2 i_3}^{(n_3-n_2)} \cdots P_{i_{N-1} i_N}^{(n_N-n_{N-1})} && \text{[By time-homogeneity]} \\ &= P\left(X_{n_1+t} = i_1\right) P_{i_1 i_2}^{(n_2-n_1)} P_{i_2 i_3}^{(n_3-n_2)} \cdots P_{i_{N-1} i_N}^{(n_N-n_{N-1})} && \text{[By assumption]} \\ &= P\left(X_{n_1+t} = i_1, X_{n_2+t} = i_2, \ldots, X_{n_N+t} = i_N\right) && \text{[By time-homogeneity]} \end{aligned}$$

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Remark. This implies that for a homogeneous DTMC, we only need to look at the time invariance of the marginals to establish stationarity!

However, this is not true for a general random process. To see this, consider the process X such that $\mathbb{P}(X_n = 1) = \mathbb{P}(X_n = -1) = 1/2, \forall n \geq 0$. However, we have that $\mathbb{P}(X_n = i, X_{n+1} = j), i, j \in \{-1, 1\}$ is distributed according to the table below:

	$X_{n+1} = -1$	$X_{n+1} = 1$
$X_n = -1$	0.3	0.2
$X_n = 1$	0.2	0.3

($n \to$ even)

	$X_{n+1} = -1$	$X_{n+1} = 1$
$X_n = -1$	0.4	0.1
$X_n = 1$	0.1	0.4

($n \to$ odd)

It is easy to see that shifts of the time-step lead to a collapse of the time-invariance of the 2-element joint. (Henceforth, DTMC $\equiv$ time-homogenous DTMC).

Exercise 11.2. 1. If $X = (X_n : n \geq 0)$ is a DTMC on a state space $\mathcal{S}$, then prove that for $n < n_1 < n_2 < \cdots < n_m$ and $i_0, i_1, \ldots, i_{n-1}, i, j_1, \ldots, j_m \in \mathcal{S}$,

$$P\left(X_{n_i} = j_i i, 1 \leq i \leq m \mid X_0 = i_0, X_1 = i_1, \ldots, X_n = i\right) = P\left(X_{n_i} = j_i, 1 \leq i \leq m \mid X_n = i\right)$$

Hint: Condition and sum.

2. Show that for any $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_{n-1} \subseteq \mathcal{S}$ and $i, j \in \mathcal{S}$

$$P\left(x_{n+1} = j \mid X_0 \in \mathcal{A}_0, X_1 \in \mathcal{A}_1, \ldots, X_{n-1} \in \mathcal{A}_{n-1}, X_n = i\right) = P\left(X_{n+1} = j \mid X_n = i\right).$$

Exercise 11.3. If the random variables X, Y, Z (in that order) obey the Markov Property, i.e., X—Y—Z is a DTMC, does Z, Y, X also have the Markov Property? That is, do we have that Z—Y—X?

11.1 An alternate view of DTMCs

We review the "Random Mapping Representation" theorem, discussed in the lectures, in this subsection.

Often, we will encounter descriptions of *systems with state* that obey update equations of the form

$$x\left(t+1\right) = f_t\left(x\left(t\right)\right),\ t \in \mathbb{N} \cup \{0\} \text{ with } x\left(0\right) = C,\ C \in \mathbb{R}.$$

The above expression is a description of a special kind of *causal* dynamical system, where the causality here referes to $x(t)$ being dependent only on the history upto that point t, and not the future.

Simple examples of such systems include the system with the position of a particle as the state $x(t)$ and an update equation based on the velocity of the particle at time $t-1$. (Try writing this down – the system description is a discrete-time approximation of the o.d.e.: $\frac{dx(t)}{dt} = v(t)$.)

Now, let us inject stochasticity into the system description – the state is now a random variable. In particular, let $U = (U_n : n \in \mathbb{N} \cup \{0\})$ be an i.i.d. sequence. Let the *state* update rule be written as

$$X_{n+1} = f\left(X_n, U_n\right)\ n \geq 0,\ \text{where } f : \mathfrak{X} \times \mathcal{U} \to \mathfrak{X}. \tag{11.1}$$

Exercise 11.4. Suppose that $U_n, n \geq 0$ takes values in a finite set $\mathcal{U}$. Can you show that $X = (X_n : n \in \mathbb{N} \cup \{0\})$ is a DTMC? Is it time-homogeneous?

If you've tried out the exercise above, you will realize that the answer to the last question is in the affirmative. In fact, the *Random Mapping Representation* theorem establishes the converse: any homogeneous DTMC can be represented by the update equation eq. (11.1), for a suitably defined defined f.

Exercise 11.5. Show that by the system form eqn. (11.1) of a DTMC

$$\begin{aligned} P\left(X_{n+1} = j \mid X_n = i\right) &= P\left(f\left(i, U_n\right) = j\right) \\ &= P\left(f\left(i, U_0\right) = j\right). \end{aligned}$$

This is another way of verifying time-homogeneity.

We conclude this discussion with a simple example.

Example 11.6. There are N empty boxes and an infinite collection of balls. At teach step, a box is chosen at random and a ball placed in it. Let X_n be the number of empty boxes afther the n^{th} ball has been placed.

1. Show that $X : \Omega \to \mathcal{X}^{\mathbb{N}}$ is a DTMC, for a suitably defined Ω.
2. What are its transition probabilities?

Solution. 1. Observe that $X_{n+1} = X_n + Z_n$, where

$$Z_n = \begin{cases} 0, & \text{w.p. } \frac{X_n}{N}, \\ -1, & \text{w.p. } \frac{N-X_n}{N}. \end{cases}$$

Hence, conditioned on X_n, $X_{n+1} \perp (X_0, X_1, \ldots, X_{n-1})$. Therefore, $(X_n : n \geq 0)$ is a DTMC.

2. Note that $X_n \in \{0, 1, 2, \ldots, N\}$. For all $n \geq 0$,

$$\begin{aligned} P\left(X_{n+1} = j+1 \mid X_n = j\right) &= 0, \\ P\left(X_{n+1} = j \mid X_n = j\right) &= \frac{j}{N}, \\ P\left(X_{n+1} = j \mid X_n = j-1\right) &= \frac{N-j}{N}, \\ \text{and } P\left(X_{n+1} = s \mid X_n = j\right) &= 0 \text{ o.w.} \end{aligned}$$

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