Tutorial-13: Invariant Distribution of Markov chains

1 Computation of invariant distribution

We will focus on finding the invariant distribution $\pi \in \mathcal{M}(\mathcal{X})$ of a time homogeneous discrete time Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$, with transition probability $P : \mathcal{X} \times \mathcal{X} \to [0,1]$ such that the xth row is the conditional distribution $P_x \in \mathcal{M}(\mathcal{X})$ with initial state $X_0 = x$.

1.1 Global balance equations

Recall that $\pi \in \mathcal{M}(\mathcal{X})$ is an invariant distribution if $\pi = \pi P$.

Example 1.1. Let the transition matrix be

$$P = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3}. \end{bmatrix}$$

We can show that P is irreducible and aperiodic. Since it is finite state Markov chain, it follows that it has a unique invariant distribution π, for which we get the set of linear equations

$$\begin{aligned} \pi_0 &= \frac{1}{4}\pi_0 + \frac{1}{3}\pi_2 \\ \pi_1 &= \frac{1}{4}\pi_0 + \frac{1}{3}\pi_1 + \frac{1}{3}\pi_2 \\ \pi_2 &= \frac{1}{2}\pi_0 + \frac{2}{3}\pi_1 + \frac{1}{3}\pi_2. \end{aligned}$$

Solving them, we get $\pi_0 = \frac{4}{9}\pi_2$ and $\pi_1 = \frac{6}{9}\pi_2$. Since $\pi_0 + \pi_1 + \pi_2 = 1$, then $\pi_2 = \frac{9}{19}$. Therefore, we can obtain the invariant distribution $\pi = (\frac{4}{19}, \frac{6}{19}, \frac{9}{19})$. Given the initial state $X_0 = 0$, what is the mean return time to state 0?

1.2 Cut balancing approach

Recall that each transition matrix can be represented by a transition graph $G = (V, E, w)$, where

(i) the set of nodes $V = \mathcal{X}$,

(ii) the set of edges $E = \{(x,y) \in \mathcal{X} \times \mathcal{X} : p_{xy} > 0\}$, and

(iii) the weight function $w : E \to [0,1]$ defined by $w(x,y) = p_{x,y}$ for all edges $(x,y) \in E$.

Definition 1.2. A **cut** of a graph $G = (V, E, w)$ is defined by a partition $(A, V \setminus A)$ of vertices. A cut determines the **cut-set** that consists of edges with one node in each partition. That is,

$$E(A) \triangleq \{(x,y) \in E : x \in A, y \in A^c\}.$$

Remark 1. For a connected graph, each cut-set determines a unique cut.

Theorem 1.3. *At stationarity of a time homogeneous Markov chain X, the probability flux balances across any cut $A \subseteq \mathcal{X}$. That is,*

$$\sum_{(x,y)\in E(A)} \pi_x p_{xy} = \sum_{(y,x)\in E(A^c)} \pi_y p_{yx}.$$

Proof. We can write the LHS of the above equation as

$$\sum_{(x,y)\in E(A)} \pi_x p_{xy} = \sum_{x\in A}\sum_{y\in A^c} \pi_x p_{xy} = \sum_{y\in A^c}\sum_{x\in \mathcal{X}} \pi_x p_{xy} - \sum_{y\in A^c}\sum_{x\in A^c} \pi_x p_{xy} = \sum_{y\in A^c} \pi_y - \sum_{y\in A^c}\sum_{x\in A^c} \pi_x p_{xy}.$$

We can write the RHS of the above equation as

$$\sum_{(y,x)\in E(A^c)} \pi_y p_{yx} = \sum_{y\in A^c} \pi_y \sum_{x\in A} p_{yx} = \sum_{y\in A^c} \pi_y \sum_{x\in \mathcal{X}} p_{yx} - \sum_{y\in A^c}\sum_{x\in A^c} \pi_y p_{yx} = \sum_{y\in A^c} \pi_y - \sum_{y\in A^c}\sum_{x\in A^c} \pi_x p_{xy}.$$

□

Example 1.4 (Birth-death processes). Consider a homogeneous DTMC $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$ with ordered state space $\mathcal{X} \subseteq \mathbb{Z}_+$ and the transition probability matrix P such that $p_{x,y} = 0$ for all $|y - x| > 1$. For each state x, we take cut $A_x = \{y \in \mathcal{X} : y \leqslant x\}$. Balancing the probability flux across the cuts, we get

$$\pi_x p_{x,x+1} = \pi_{x+1} p_{x+1,x}.$$

For example, consider the state space $\mathcal{X} = \mathbb{Z}_+$ and for some $0 \leqslant \alpha < \beta < 1 - \alpha$. If the birth-death process has transition probability matrix P such that $p_{x,x+1} = \alpha$ for all $x \geqslant 0$, and $p_{x,x-1} = \beta$ for all $x \in \mathbb{N}$ and $p_{0,0} = 1 - \alpha$. Then, we observe that

$$\pi_x = \left(\frac{\alpha}{\beta}\right)^x \pi_0, \quad x \in \mathbb{N}.$$

Since $\sum_{x\in\mathbb{Z}_+} \pi_x = 1$, we get $\pi_0 = \frac{1}{1-\frac{\alpha}{\beta}}$.

1.3 Transform approach

Definition 1.5. Each distribution $\nu \in \mathcal{M}(\mathbb{Z}_+)$ is completely determined by its z-transform defined by

$$\Psi_\nu(z) \triangleq \sum_{x\in\mathbb{Z}_+} z^x \nu_x, \quad |z| < 1.$$

From the global balance equation for a homogeneous Markov chain X with state space $\mathbb{Z}_+$, we can write its z-transform as

$$\Psi_\pi(z) = \sum_{y\in\mathbb{Z}_+} \pi_y z^y = \sum_{y\in\mathbb{Z}_+} z^y \sum_{x\in\mathbb{Z}_+} \pi_x p_{xy} = \sum_{x\in\mathbb{Z}_+} \pi_x \Psi_{P_x}(z).$$

In some cases, it is easy to compute $\Psi_{P_x}(z)$, and we will be able to get an explicit z-transform for $\Psi_\pi(z)$ which we will be able to invert to get the invariant distribution π

Example 1.6 (Homogeneous birth-death processes). For $x = 0$, we have $P_x = (1 - \alpha, \alpha, 0, \dots)$. For $x \in \mathbb{N}$, we have $P_x = \beta e_{x-1} + (1 - \alpha - \beta)e_x + \alpha e_{x+1}$. Therefore, $\Psi_{P_x}(z) = (1 - \alpha) + \alpha z$ for $x = 0$ and

$$\Psi_{P_x}(z) = \beta z^{x-1} + (1 - \alpha - \beta) z^x + \alpha z^{x+1}, \quad x \in \mathbb{N}.$$

Therefore, we can write

$$\begin{aligned}\Psi_\pi(z) &= \sum_{x\in\mathbb{Z}_+} \pi_x \Psi_{P_x}(z) = \pi_0(1 - \alpha + \alpha z) + \sum_{x\in\mathbb{N}} \pi_x(\beta z^{x-1} + (1 - \alpha - \beta) z^x + \alpha z^{x+1}) \\ &= (1 - \alpha - \beta)\Psi_\pi(z) + \beta\pi_0 + \alpha z \Psi_\pi(z) + z^{-1}\beta(\Psi_\pi(z) - \pi_0).\end{aligned}$$

Aggregating these results, we get

$$\Psi_\pi(z) = \pi_0 \frac{\beta(z-1)}{\beta(z-1) - \alpha z(z-1)} = \pi_0 \frac{1}{1 - \frac{\alpha}{\beta} z}.$$

Inverting the z-transform for $\alpha < \beta$, we get

$$\pi_x = \pi_0 \left(\frac{\alpha}{\beta}\right)^x, \quad x \in \mathbb{Z}_+.$$