Lecture-03: Stochastic Processess

1 Stochastic Processes

Definition 1.1 (Random process). Let $(\Omega, \mathcal{F}, P)$ be a probability space. For an arbitrary index set T and state space $\mathcal{X} \subseteq \mathbb{R}$, a **random process** is a measurable map $X : \Omega \to \mathcal{X}^T$. That is, for each outcome $\omega \in \Omega$, we have a function $X(\omega) : T \mapsto \mathcal{X}$ called the **sample path** or the **sample function** of the process X, also written as

$$X(\omega) \triangleq (X_t(\omega) \in \mathcal{X} : t \in T).$$

1.1 Classification

State space $\mathcal{X}$ can be countable or uncountable, corresponding to discrete or continuous valued process. If the index set T is countable, the stochastic process is called **discrete**-time stochastic process or random sequence. When the index set T is uncountable, it is called **continuous**-time stochastic process. The index set T doesn't have to be time, if the index set is space, and then the stochastic process is spatial process. When $T = \mathbb{R}^n \times [0, \infty)$, stochastic process X is a spatio-temporal process.

1.2 Measurability

For any finite subset $S \subseteq T$ and real vector $x \in \mathbb{R}^T$ such that $x_t = \infty$ for any $t \notin S$, we define a set

$$A(x) \triangleq \left\{ y \in \mathbb{R}^T : y_t \leqslant x_t \right\} = \bigtimes_{t \in T} (-\infty, x_t] = \bigtimes_{s \in S} (-\infty, x_s] \bigtimes_{t \notin S} \mathbb{R}.$$

Then, the measurability of the random process X implies that for any such set $A(x)$, we have

$$X^{-1}(A(x)) = X^{-1} \bigtimes_{t \in T} (-\infty, x_t] = \cap_{t \in T} X_t^{-1}(-\infty, x_t] = \cap_{s \in S} X_s^{-1}(-\infty, x_s] \in \mathcal{F}.$$

Remark 1. Realization of random process at each $t \in T$, is a random variable defined on the probability space $(\Omega, \mathcal{F}, P)$ such that $X_t : \Omega \to \mathcal{X}$. This follows from the fact that for any $t \in T$ and $x_t \in \mathbb{R}$, we can take Borel measurable sets $A(x) = \bigtimes(-\infty, x_t] \bigtimes_{s \neq t} \mathbb{R}$. Then, $X^{-1}(A(x)) = X_t^{-1}(-\infty, x_t] \in \mathcal{F}$.

Remark 2. The random process X can be thought of as a collection of random variables $X : T \to \mathcal{X}^\Omega$ or an ensemble of sample paths $X : \Omega \to \mathcal{X}^T$. Recall that $\mathcal{X}^T$ is set of all functions from the index set T to state space $\mathcal{X}$.

1.3 Distribution

To define a measure on a random process, we can either put a measure on sample paths, or equip the collection of random variables with a joint measure. We are interested in identifying the joint distribution $F : \mathbb{R}^T \to [0, 1]$. To this end, for any $x \in \mathbb{R}^T$ we need to know

$$F_X(x) \triangleq P\left(\bigcap_{t \in T} \{\omega \in \Omega : X_t(\omega) \leqslant x_t\} \right) = P(\bigcap_{t \in T} X_t^{-1}(-\infty, x_t]) = P \circ X^{-1} \bigtimes_{t \in T} (-\infty, x_t].$$

However, even for a simple independent process with countably infinite T, any function of the above form would be zero if x_t is finite for all $t \in T$. Therefore, we only look at the values of $F(x)$ when $x_t \in \mathbb{R}$ for indices t in a finite set S and $x_t = \infty$ for all $t \notin S$. That is, for any finite set $S \subseteq T$, we focus on the product sets of the form

$$A(x) \triangleq \bigtimes_{s \in S} (-\infty, x_s] \bigtimes_{s \notin S} \mathbb{R},$$

where $x \in \mathcal{X}^T$ and $x_t = \infty$ for $t \notin S$. Recall that by definition of measurability, $X^{-1}(A(x)) \in \mathcal{F}$, and hence $P \circ X^{-1}(A(x))$ is well defined.

Definition 1.2. We can define a **finite dimensional distribution** for any finite set $S \subseteq T$ and $x_S = \{x_s \in \mathbb{R} : s \in S\}$,

$$F_{X_S}(x_S) \triangleq P\left(\bigcap_{s\in S}\{\omega \in \Omega : X_s(\omega) \leqslant x_s\}\right) = P(\bigcap_{s\in S} X_s^{-1}(-\infty, x_s]).$$

Set of all finite dimensional distributions of the stochastic process $X = (X_t \in \mathcal{X}^\Omega : t \in T)$ characterizes its distribution completely. Simpler characterizations of a stochastic process X are in terms of its moments. That is, the first moment such as mean, and the second moment such as correlations and covariance functions.

$$m_X(t) \triangleq \mathbb{E}X_t, \qquad R_X(t,s) \triangleq \mathbb{E}X_tX_s, \qquad C_X(t,s) \triangleq \mathbb{E}(X_t - m_X(t))(X_s - m_X(s)).$$

Example 1.3 (Bernoulli sequence). Let index set $T = \mathbb{N} = \{1,2,\dots\}$ and the sample space be the collection of infinite bi-variate sequences of successes (S) and failures (F) defined by $\Omega = \{S,F\}^{\mathbb{N}}$. An outcome $\omega \in \Omega$ is an infinite sequence $\omega = (\omega_1, \omega_2, \dots)$ such that $\omega_n \in \{S,F\}$ for each $n \in \mathbb{N}$. We let the event space $\mathcal{F} = \sigma(E_n : n \in \mathbb{N})$ where E_n is the event of first appearance of success at the nth outcome.

We define the random process $X : \Omega \to \{0,1\}^{\mathbb{N}}$ such that $X(\omega) = (\mathbb{1}_{\{S\}}(\omega_1), \mathbb{1}_{\{S\}}(\omega_2), \dots)$. That is, we have

$$X_n(\omega) = \mathbb{1}_{\{S\}}(\omega_n), \qquad X(\omega) = (\mathbb{1}_{\{S\}}(\omega_n) : n \in \mathbb{N}).$$

Hence, we can write the process as collection of random variables $X = (X_n \in \{0,1\}^\Omega : n \in \mathbb{N})$ or the collection of sample paths $X = (X(\omega) \in \{0,1\}^{\mathbb{N}} : \omega \in \Omega)$.

1.4 Independence

A stochastic process X is said to be **independent** if for all finite subsets $S \subseteq T$, the finite collection of events $\{\{X_s \leqslant x_s\} : s \in S\}$ are independent. That is, we have

$$F_{X_S}(x_S) = P(\cap_{s\in S}\{X_s \leqslant x_s\}) = \prod_{s\in S} P\{X_s \leqslant x_s\} = \prod_{s\in S} F_{X_s}(x_s).$$

Two stochastic process X, Y for the common index set T are **independent random processes** if for all finite subsets $I, J \subseteq T$, the following events $\{X_i \leqslant x_i, i \in I\}$ and $\{Y_j \leqslant y_j, j \in J\}$ are independent. That is,

$$F_{X_I,Y_J}(x_I, y_J) \triangleq P\left(\{X_i \leqslant x_i, i \in I\} \cap \{Y_j \leqslant y_j, j \in J\}\right) = P\left(\cap_{i\in I}\{X_i \leqslant x_i\}\right) P\left(\cap_{j\in J}\{Y_j \leqslant y_j\}\right) = F_{X_I}(x_I) F_{Y_J}(y_J).$$

Example 1.4 (Bernoulli sequence). Let the Bernoulli sequence X defined in Example 1.3 be independent and identically distributed with $P\{X_n = 1\} = p \in (0,1)$. For any sequence $x \in \{0,1\}^{\mathbb{N}}$, we have $P\{X = x\} = 0$. Let $q \triangleq (1-p)$, then the probability of observing m heads and r tails is given by $p^m q^r$. We can easily compute the mean, the auto-correlation, and the auto-covariance functions for the independent Bernoulli process defined in Example 1.4 as

$$m_X(n) = \mathbb{E}X_n = p, \qquad R_X(m,n) = \mathbb{E}X_mX_n = \mathbb{E}X_m\mathbb{E}X_n = p^2, \qquad C_x(m,n) = 0.$$

1.5 Filtration

Definition 1.5. A net of σ-algebras $\mathcal{F}_\bullet = \{\mathcal{F}_t \subseteq \mathcal{F} : t \in T\}$ is called a **filtration** when the index set T is totally ordered and the net is non-decreasing, that is for all $s \leqslant t$ we have $\mathcal{F}_s \subseteq \mathcal{F}_t$.

Definition 1.6. Consider a real-valued random process X indexed by the ordered set T on the probability space $(\Omega, \mathcal{F}, P)$. The process X is called **adapted** to the filtration $\mathcal{F}_\bullet$, if for each $t \in T$, we have $\sigma(X_t) \subseteq \mathcal{F}_t$ or $X_t^{-1}(-\infty, x] \in \mathcal{F}_t$ for each $x \in \mathbb{R}$.

Definition 1.7. We can define a natural filtration $\mathcal{F}_\bullet = \{\mathcal{F}_t \subseteq \mathcal{F} : t \in T\}$ indexed by totally ordered T for the random process $X = (X_s : s \in T)$, where $\mathcal{F}_t \triangleq \sigma(X_s, s \leqslant t)$ is the information about the process until index t and the process X is adapted to its natural filtration by definition.

Remark 3. If $X = (X_t : t \in T)$ is an independent process with the associated natural filtration $\mathcal{F}_\bullet$, then for any $t > s$ and events $A \in \mathcal{F}_s$, the random variable X_t is independent of the event A. This is just a fancy way of saying X_t is independent of $(X_u, u \leqslant s)$. Hence, for any random variable $Y \in \mathcal{F}_s$, we have

$$\mathbb{E}[\mathbb{E}[X_t Y | \mathcal{F}_s]] = \mathbb{E}[\mathbb{E}[X_t] Y] = \mathbb{E}X_t \mathbb{E}Y.$$