Lecture-04: Stopping Times

1 Stopping Times

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $\mathcal{F}_\bullet = (\mathcal{F}_t \subseteq \mathcal{F} : t \in T)$ be a filtration on this probability space for an ordered index set $T \subseteq \mathbb{R}$.

Definition 1.1. An almost surely finite random variable $\tau : \Omega \to T$ defined on this probability space is called a **stopping time** with respect to this filtration if the event

$$\{\tau \leqslant t\} \in \mathcal{F}_t \text{ for all } t \in T.$$

Remark 1. We can consider the ordered index set T as time. For a real-valued time-evolving random process $X : \Omega \to \mathcal{X}^T$ defined on this space, let $\mathcal{F}_\bullet$ be its natural filtration, i.e. $\mathcal{F}_t = \sigma(X_s, s \leqslant t)$ for all times $t \in T$. A stopping time $\tau : \Omega \to T$ for the process X is a random variable such that if we observe the process X sequentially, then the event $\{\tau \leqslant t\}$ can be completely determined by the sequential observation $(X_s, s \leqslant t)$ until time t.

Remark 2. For the special case when $I = \mathbb{N}$ is a countable ordered index set, the stopping time can be defined as a random variable $N : \Omega \to \mathbb{N} \cup \{\infty\}$ taking countably many values, if $P(\{N \in \mathbb{N}\}) = 1$ and the event $\{N = n\} \in \mathcal{F}_n$ for each $n \in \mathbb{N}$. This follows from the fact that $\{N = n\} = \{N \leqslant n\} \cap \{N \leqslant n-1\}^c \in \mathcal{F}_n$ and $\cup_{m \leqslant n} \{N = m\} = \{N \leqslant n\}$.

Example 1.2. Examples of stopping times.

1. While traveling on the bus, consider the random process as the bus stops. The random variable measuring "time until bus crosses next stop after Majestic" is a stopping time as it's value is determined by events before it happens. On the other hand "time until bus crosses the stop before Majestic" would not be a stopping time in the same context. This is because we have to cross this stop, reach Majestic and then realize we have crossed that point.

2. Consider an *iid* Bernoulli sequence $X : \Omega \to \{0,1\}^{\mathbb{N}}$, and define the number of successes until time n as $S_n \triangleq \sum_{i=1}^n X_i$. The following are stopping times,

$$T_k \triangleq \min\{n \in \mathbb{N} : S_n = k\}, \text{ for all } k \in \mathbb{N}.$$

   We will show that T_k is almost surely finite. We can verify that $\{T_k = n\} = \cap_{i=1}^{n-1} \{S_i < k\} \cap \{S_n = k\} \in \mathcal{F}_n$ for each $n \in \mathbb{N}$.

1.1 Properties of stopping time

Lemma 1.3. *Let $\tau_1, \tau_2 : \Omega \to T$ be stopping times on probability space $(\Omega, \mathcal{F}, P)$ with respect to filtration $\mathcal{F}_\bullet = (\mathcal{F}_t, t \in T)$. Then the following hold true.*

*i_* $\min\{\tau_1, \tau_2\}$ *and* $\max \tau_1, \tau_2$ *are stopping times.*

ii_ If $P(\{\tau \in I\}) = 1$ for a countable $I \subseteq T$, then $\tau_1 + \tau_2$ is a stopping time.

Proof. Let $\mathcal{F}_\bullet = (\mathcal{F}_t : t \in T)$ be a filtration, and τ_1, τ_2 associated stopping times.

i_ Result follows since the event $\{\min\{\tau_1, \tau_2\} > t\} = \{\tau_1 > t\} \cap \{\tau_2 > t\} \in \mathcal{F}_t$, and the event $\{\max\{\tau_1, \tau_2\} \leqslant t\} = \{\tau_1 \leqslant t\} \cap \{\tau_2 \leqslant t\} \in \mathcal{F}_t$ for any time $t \in T$.

ii_ It suffices to show that the event $\{\tau_1 + \tau_2 \leqslant t\} \in \mathcal{F}_t$ for $t \in I = \mathbb{N}$. To this end, we observe that

$$\{\tau_1 + \tau_2 \leqslant n\} = \bigcup_{m \in \mathbb{N}} \{\tau_1 \leqslant n - m, \tau_2 \leqslant m\} \in \mathcal{F}_n.$$

$\square$

Lemma 1.4 (Wald's lemma). *Consider a random walk $S : \Omega \to \mathbb{R}^{\mathbb{N}}$ with iid step-sizes $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ having finite $\mathbb{E}|X_1|$, Let $N : \Omega \to \mathbb{N}$ be a finite mean stopping time adapted to the natural filtration $\mathcal{F}_\bullet = (\mathcal{F}_n \triangleq \sigma(X_1,\dots,X_n) : n \in \mathbb{N})$ Then,*

$$\mathbb{E}S_N = \mathbb{E}X_1\mathbb{E}N.$$

Remark 3. Recall that when N is independent of the random sequence X, the similar result holds. The proof is really simple, as we can write

$$\mathbb{E}S_N = \mathbb{E}[\mathbb{E}[S_N|N]] = \mathbb{E}[\mathbb{E}[S_N\mathbb{1}_\Omega|N]] = \mathbb{E}[\sum_{n\in\mathbb{N}}\mathbb{E}[S_N\mathbb{1}_{\{N=n\}}|N]] = \mathbb{E}[\sum_{n\in\mathbb{N}}\mathbb{1}_{\{N=n\}}\mathbb{E}[S_N|N=n]] = \sum_{n\in\mathbb{N}}\mathbb{E}\mathbb{1}_{\{N=n\}}\sum_{i=1}^{n}\mathbb{E}[X_i|N=n].$$

Since the random sequence X and random variable N are independent, we have $\mathbb{E}[X_i|N=n] = \mathbb{E}X_i$. Since the sequence X is also identical, we get

$$\mathbb{E}S_N = \mathbb{E}X_1\sum_{n\in\mathbb{N}} nP\{N=n\} = \mathbb{E}X_1\mathbb{E}N.$$

Remark 4. Let's examine why this proof breaks down when N is a stopping time with respect to natural filtration of X. In this case, it is not clear what is the value $\mathbb{E}[X_i|N=n]$? For example, consider the *iid* sequence $X \in \{0,1\}^{\mathbb{N}}$ with $P(X_i = 1) = p$ and stopping $N \triangleq \inf\{n \in \mathbb{N} : X_i = 1\}$ adapted to natural filtration of X. In this case,

$$\mathbb{E}[X_i|N=n] = \mathbb{1}_{\{i=n\}} \neq \mathbb{E}X_i = p.$$

However, we do notice that the result somehow *magically* continues to hold, as

$$\mathbb{E}S_N = \sum_{n\in\mathbb{N}}\mathbb{E}\mathbb{1}_{\{N=n\}} = 1 = \mathbb{E}X_1\mathbb{E}N = \frac{p}{p}.$$

Proof. From the independence of step sizes, it follows that X_n is independent of $\mathcal{F}_{n-1}$. Next we observe that $\{N \geqslant n\} = \{N > n-1\} \in \mathcal{F}_{n-1}$, and hence $\mathbb{E}[X_n 1_{\{N\geqslant n\}}] = \mathbb{E}X_n\mathbb{E}1_{\{N\geqslant n\}}$. Therefore,

$$\mathbb{E}\sum_{n=1}^{N} X_n = \mathbb{E}\sum_{n\in\mathbb{N}} X_n 1_{\{N\geqslant n\}} = \sum_{n\in\mathbb{N}}\mathbb{E}X_n\mathbb{E}\left[1_{\{N\geqslant n\}}\right] = \mathbb{E}X_1\mathbb{E}\left[\sum_{n\in\mathbb{N}} 1_{\{N\geqslant n\}}\right] = \mathbb{E}[X_1]\mathbb{E}[N]. \tag{1}$$

We exchanged limit and expectation in the above step, which is not always allowed. We were able to do it since the summand is positive and we apply monotone convergence theorem. $\square$

1.2 Stopping time σ-algebra

We wish to define an event space consisting information of the process until a random time τ. For a stopping time $\tau : \Omega \to T$, what we want is something like $\sigma(X_s : s \leqslant \tau)$. But that doesn't make sense, since the random time τ is a random variable itself. When τ is a stopping time, the event $\{\tau \leqslant t\} \in \mathcal{F}_t$. What makes sense is the set of all events whose intersection with $\{\tau \leqslant t\}$ belongs to the event subspace $\mathcal{F}_t$ for all $t \geqslant 0$.

Definition 1.5. For a stopping time $\tau : \Omega \to T$ adapted to the filtration $\mathcal{F}_\bullet$, the **stopping time σ-algebra** is defined

$$\mathcal{F}_\tau \triangleq \{A \in \mathcal{F} : A \cap \{\tau \leqslant t\} \in \mathcal{F}_t, \text{ for all } t \in T\}.$$

We will first show that $\mathcal{F}_\tau$ is indeed a σ-algebra.

1. Since τ is a stopping time, it follows that $\Omega \in \mathcal{F}_\tau$. Further, since $\emptyset \in \mathcal{F}_t$, we have $\emptyset \in \mathcal{F}_\tau$.
2. From closure of $\mathcal{F}_t$ under countable unions, it follows that $\mathcal{F}_\tau$ is closed under countable unions.
3. Let $A \in \mathcal{F}_\tau$, then $A \cap \{\tau \leqslant t\} \in \mathcal{F}_t$ and we can write $A^c \cap \{\tau \leqslant t\} = \{\tau \leqslant t\} \setminus (A \cap \{\tau \leqslant t\}) \in \mathcal{F}_t$.

Informally, the event space $\mathcal{F}_\tau$ has information up to the random time τ. That is, it is a collection of measurable sets that are determined by the process until time τ. Any measurable set $A \in \mathcal{F}$ can be written as $A = (A \cap \{\tau \leqslant t\}) \cup (A \cap \{\tau > t\})$. All such sets A such that $A \cap \{\tau \leqslant t\} \in \mathcal{F}_t$ is a member of the stopped σ-algebra.

Lemma 1.6. *Let τ, τ_1, τ_2 be stopping times, and $X : \Omega \to \mathcal{X}^T$ a random process, all adapted to a filtration $\mathcal{F}_\bullet = (\mathcal{F}_t, t \in T)$. Then, the following are true.*

i_ If $\tau_1 \leqslant \tau_2$ almost surely, then $\mathcal{F}_{\tau_1} \subseteq \mathcal{F}_{\tau_2}$.

*ii_* $\sigma(\tau) \subseteq \mathcal{F}_\tau$, *and* $\sigma(X_\tau) \subseteq \mathcal{F}_\tau$.

Proof. Recall, that for any $t \geqslant 0$, we have $\{\tau \leqslant t\} \in \mathcal{F}_t$.

i_ From the hypothesis $\tau_1 \leqslant \tau_2$ a.s., we get $\{\tau_2 \leqslant t\} \subseteq \{\tau_1 \leqslant t\}$ a.s., where both events belong to $\mathcal{F}_t$ since they are stopping times. The result follows since for any $A \in \mathcal{F}_{\tau_1}$, we can write $A \cap \{\tau_2 \leqslant t\} = A \cap \{\tau_1 \leqslant t\} \cap \{\tau_2 \leqslant t\} \in \mathcal{F}_t$ for all $t \in T$.

ii_ Any event $A \in \sigma(\tau)$ is generated by inverse images $\{\tau \leqslant s\}$ for $s \in \mathbb{R}$. Indeed $\{\tau \leqslant s\} \in \mathcal{F}_\tau$ since $\{\tau \leqslant s\} \cap \{\tau \leqslant t\} = \{\tau \leqslant s \wedge t\} \in \mathcal{F}_t$, for all $t \in T$.

The events of the form $\{X_\tau \leqslant x\}$ for real $x \in \mathbb{R}$ generate the event subspace $\sigma(X_\tau)$, and event $\{X_\tau \leqslant x\} \cap \{\tau \leqslant t\} \in \mathcal{F}_t$ for all $t \in T$. This implies that $\sigma(X_\tau) \subseteq \mathcal{F}_\tau$.

$\square$

Lemma 1.7. *Let $\mathcal{F}_\bullet$ be the natural filtration associated with the process $X : \Omega \to \mathcal{X}^T$, and τ be an associated stopping time. Let $\mathcal{H} \triangleq \sigma(X_{\tau \wedge t}, t \in T)$ be the event space generated by the stopped process $(X_{\tau \wedge t} : t \in T)$ and $\mathcal{F}_\tau$ be the stopping-time event space. Then $\mathcal{F}_\tau = \mathcal{H}$ for T discrete.*

Proof. Let $A \in \mathcal{H}$, then we have $A \cap \{\tau \leqslant t\} \in \mathcal{F}_t$ for any $t \in T$, and hence $\mathcal{H} \subseteq \mathcal{F}_\tau$. For the converse, we assume $T = \mathbb{N}$ and we need to show that for any $A \in \mathcal{F}_\tau$ we have $A \cap \{\tau = k\} \in \mathcal{H}$ for all $k \in \mathbb{N}$. We will show this by induction on $k \in \mathbb{N}$.

$k = 1$: We take any $A \in \mathcal{F}_\tau$, then $A \cap \{\tau = 1\} \in \mathcal{F}_1 \subseteq \mathcal{H}$ since $\tau \geqslant 1$ almost surely.

$k > 1$: We assume that the induction hypothesis holds for some $k - 1 \in \mathbb{N}$. For any $A \in \mathcal{F}_\tau$, we have $A \cap \{\tau = k\} \in \mathcal{F}_k = \sigma(X_1, \dots, X_k)$. Further, $\{\tau = k\} = \{\tau = k\} \cap \{\tau \leqslant k\}$, and therefore, we can write

$$\mathbb{1}_{A \cap \{\tau = k\}} = f(X_1, \dots, X_k)\mathbb{1}_{\{\tau \geqslant k\}} = f(X_{\tau \wedge 1}, \dots, X_{\tau \wedge k})(1 - \mathbb{1}_{\{\tau \leqslant k-1\}}) \in \mathcal{H}.$$

This implies that $A \cap \{\tau = k\} \in \mathcal{H}$, and hence $\mathcal{F}_\tau \subseteq \mathcal{H}$.

$\square$