Lecture-05: Strong Markov Property

1 Strong Markov property

Definition 1.1. A process $X : \Omega \to \mathcal{X}^{\mathbb{R}}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ adapted to its natural filtration $\mathcal{F}_\bullet = (\mathcal{F}_t \triangleq \sigma(X_s, s \leqslant t), t \in T)$, is called **Markov** if we have we have

$$P(\{X_t \leqslant x\} | \mathcal{F}_s) = P(\{X_t \leqslant x\} | \sigma(X_s)).$$

Example 1.2. An *iid* process is trivially Markov, since

$$P(\{X_t \leqslant x\} | \mathcal{F}_s) = P(\{X_t \leqslant x\} | \sigma(X_s)) = P(\{X_t \leqslant x\}).$$

Example 1.3. Let $X : \Omega \to \mathcal{X}^{\mathbb{N}}$ be an iid process, and a random walk process $S : \Omega \to \mathcal{X}^{\mathbb{N}}$ is defined in term of step-size process X as $S_n \triangleq \sum_{i=1}^n X_i$ for all $n \in \mathbb{N}$. Then the random walk S is Markov with respect to its natural filtration $\mathcal{F}_\bullet$. To see this, take $n \in \mathbb{N}$ and observe from the independence of X_{n+1} and $\mathcal{F}_n$ that

$$P(\{S_{n+1} \leqslant x\} | \mathcal{F}_n) = P(\{X_{n+1} \leqslant x - S_n\}) = P(\{X_{n+1} \leqslant x - S_n\} | \sigma(S_n)) = P(\{S_{n+1} \leqslant x\} | \sigma(S_n)).$$

Definition 1.4. Let $X : \Omega \to \mathcal{X}^T$ be a real valued Markov process adapted to a filtration $\mathcal{F}_\bullet$. Let τ be an almost surely finite stopping time with respect to this filtration, then the process X is called **strongly Markov** if for all $x \in \mathbb{R}$ and $t > 0$, we have

$$P(\{X_{t+\tau} \leqslant x\} | \mathcal{F}_\tau) = P(\{X_{t+\tau} \leqslant x\} | \sigma(X_\tau)).$$

Lemma 1.5. *Let $X : \Omega \to \mathcal{X}^T$ be any Markov process adapted to a filtration $\mathcal{F}_\bullet$. For any almost surely finite stopping time τ with respect to this filtration that takes only countably many values $I \subseteq T$, Markov process X is strongly Markov at this stopping time τ.*

Proof. Let $I \subseteq T$ be the countable set such that $P\{\tau \notin I\} = 0$. Fix $x \in \mathbb{R}$, then we will show that $\mathbb{E}[\mathbb{1}_{\{X_{\tau+i} \leqslant x\}} | \mathcal{F}_\tau] = \mathbb{E}[\mathbb{1}_{\{X_{\tau+i} \leqslant x\}} | \sigma(X_\tau)]$ for all $i > 0$. Since $\mathbb{E}[\mathbb{1}_{\{X_{\tau+i} \leqslant x\}} | \sigma(X_\tau)] \in \mathcal{F}_\tau$, it suffices to show that

$$\mathbb{E}[\mathbb{1}_A \mathbb{E}[\mathbb{1}_{\{X_{\tau+i} \leqslant x\}} | \sigma(X_\tau)]] = \mathbb{E}[\mathbb{1}_A \mathbb{1}_{\{X_{\tau+i} \leqslant x\}}], \quad \text{for all } A \in \mathcal{F}_\tau.$$

Let $A \in \mathcal{F}_\tau$, then $A \cap \{\tau = i\} \in \mathcal{F}_i$ for all $i \in I$. Then, from almost sure finiteness of τ, we can write $A = \cup_{i \in I} A \cap \{\tau = i\}$. From the tower property of conditional expectation and $\mathcal{F}_i$-measurability of $A \cap \{\tau = i\}$, we have

$$\mathbb{E}[\mathbb{1}_A \mathbb{1}_{\{X_{t+\tau} \leqslant x\}}] = \sum_{i \in I} \mathbb{E}[\mathbb{1}_{A \cap \{X_{t+\tau} \leqslant x\} \cap \{\tau = i\}}] = \sum_{i \in I} \mathbb{E}[\mathbb{E}[\mathbb{1}_{A \cap \{X_{t+i} \leqslant x\} \cap \{\tau = i\}} | \mathcal{F}_i]] = \sum_{i \in I} \mathbb{E}[\mathbb{1}_{A \cap \{\tau = i\}} \mathbb{E}[\mathbb{1}_{\{\{X_{t+i} \leqslant x\}\}} | \mathcal{F}_i]].$$

From Markov property of process X, we have $\mathbb{E}[\mathbb{1}_{\{\{X_{t+i} \leqslant x\}\}} | \mathcal{F}_i] = \mathbb{E}[\mathbb{1}_{\{\{X_{t+i} \leqslant x\}\}} | \sigma(X_i)]$. Further, $\mathbb{1}_{A \cap \{\tau = i\}} = \mathbb{1}_A \mathbb{1}_{\{\tau = i\}}$, and therefore

$$\mathbb{E}[\mathbb{1}_A \mathbb{1}_{\{X_{t+\tau} \leqslant x\}}] = \mathbb{E}[\mathbb{1}_A \sum_{i \in I} \mathbb{1}_{\{\tau = i\}} \mathbb{E}[\mathbb{1}_{\{X_{t+i} \leqslant x\}} | \sigma(X_i)]] = \mathbb{E}[\mathbb{1}_A \mathbb{E}[\mathbb{1}_{\{X_{t+\tau} \leqslant x\}} | \sigma(X_\tau)]].$$

□

Corollary 1.6. *Any Markov process on countable index set T is strongly Markov.*

Proof. For a countable index set T, all associated stopping times assume at most countably many values. □

Corollary 1.7. *Let τ be an almost surely finite stopping time with respect to the natural filtration $\mathcal{F}_\bullet$ of an iid random sequence X. Then $(X_{\tau+1}, \dots, X_{\tau+n})$ is independent of $\mathcal{F}_\tau$ for each $n \in N$ and identically distributed to $(X_1, \dots, X_n)$.*

Theorem 1.8. *Let $X : \Omega \to \mathcal{X}^T$ be any real-valued Markov process adapted to the filtration $\mathcal{F}_\bullet$, with right-continuous sample paths. If the map $t \mapsto \mathbb{E}[f(X_s)|\sigma(X_t)]$ is right-continuous for each bounded continuous function f, then X is strongly Markov.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function, $t \geqslant 0$, and τ be an $\mathcal{F}_\bullet$-adapted stopping time. It suffices to show that $f(X_t)$ satisfies the strong Markov property. For each $m \in \mathbb{N}$, consider the intervals $I_{k,m} \triangleq ((k-1)2^{-m}, k2^{-m}]$ for all $k \in [2^{2m}]$, and define

$$\tau_m \triangleq \sum_{k=1}^{2^{2m}} k2^{-m} \mathbb{1}_{\{\tau \in I_{k,m}\}}.$$

Clearly the stopping time $\tau_m \leqslant 2^m$ a.s. and takes countable values for each m. Further, τ_m is decreasing in m. From a.s. finiteness of stopping time τ, there exists an $m_0 \in \mathbb{N}$ such that $\tau_m \downarrow \tau$ for all outcomes $\omega \in \Omega$ for $m \geqslant m_0$. In addition, $\tau \leqslant \tau_m$ and hence $\mathcal{F}_\tau \subseteq \mathcal{F}_{\tau_m}$ for all $m \geqslant m_0$. From strong Markov property for countably valued stopping times, we have for each $A \in \mathcal{F}_\tau$, we have

$$\mathbb{E}[\mathbb{1}_A f(X_{\tau_m+t})] = \mathbb{E}[\mathbb{1}_A \mathbb{E}[f(X_{\tau_m+t})|\sigma(X_{\tau_m})].$$

Applying dominated convergence theorem, taking limit as $\tau_m \downarrow \tau$ on both sides, we have

$$\mathbb{E}[\mathbb{1}_A f(X_{\tau+t})] = \mathbb{E}[\mathbb{1}_A \mathbb{E}[f(X_{\tau+t})|\sigma(X_\tau)]].$$

□