Lecture-06: Renewal Process

1 Counting processes

Definition 1.1. A stochastic process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ is a **counting process** if

i_ $N(0) = 0$, and

ii_ for each $\omega \in \Omega$, the map $t \mapsto N(t)$ is non-decreasing, integer valued, and right continuous.

Lemma 1.2. *A counting process has finitely many jumps in a finite interval* $(0,t]$.

Definition 1.3. A simple counting process has discontinuities of unit size.

Definition 1.4. The nth point of discontinuity of a simple counting process $N(t)$ is called the n**th arrival instant** and denoted by $S_n : \Omega \to \mathbb{R}_+$ such that

$$S_0 \triangleq 0, \qquad S_n \triangleq \inf\{t \geqslant 0 : N(t) \geqslant n\},\ n \in \mathbb{N}.$$

The random sequence of arrival instants is denoted by $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$. The **inter arrival time** between $(n-1)$th and nth arrival is denoted by $X_n \triangleq S_n - S_{n-1}$.

Remark 1. The arrival sequence S is non-decreasing for each outcome $\omega \in \Omega$. That is, $S_n \leqslant S_{n+1}$ for all $n \in \mathbb{N}$.

Remark 2. Let $\mathcal{F}_\bullet = (\mathcal{F}_s : s \geqslant 0)$ be the natural filtration associated with the counting process N, that is $\mathcal{F}_t = \sigma(N(t), t \geqslant 0)$. Then $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is a sequence of stopping times with respect to $\mathcal{F}_\bullet$.

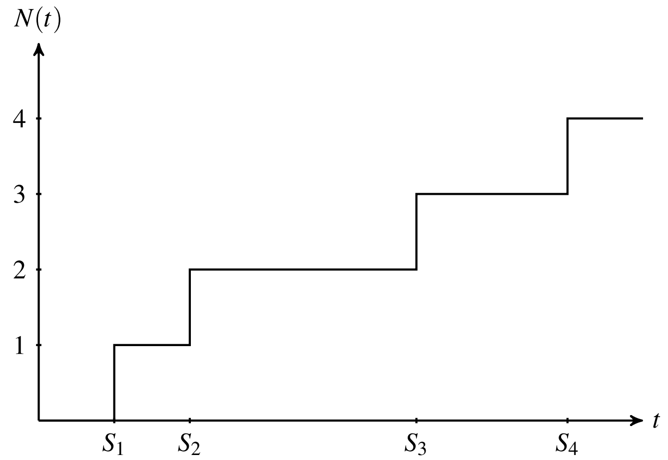

Figure 1: Sample path of a simple counting process.

Lemma 1.5 (Inverse processes). *Inverse of a simple counting process N is its corresponding arrival process S. That is,*

$$\{S_n \leqslant t\} = \{N(t) \geqslant n\}. \tag{1}$$

Proof. Let $\omega \in \{S_n \leqslant t\}$. Since N is a non-decreasing process, we have $N(t) \geqslant N(S_n) = n$. Conversely, let $\omega \in \{N(t) \geqslant n\}$, then it follows from definition that $S_n \leqslant t$. □

Corollary 1.6. *The probability mass function for the counting process $N(t)$ at time t can be written in terms of distribution functions of arrival sequence S as*

$$P\{N(t) = n\} = F_{S_n}(t) - F_{S_{n+1}}(t).$$

Proof. The event $\{N(t) \geqslant n\}$ equal union of two disjoint events $\{N(t) = n\} \cup \{N(t) \geqslant n+1\}$, and the result follows from the probability of disjoint unions. □

Definition 1.7. A **point process** is a collection $S : \Omega \to \mathcal{X}^{\mathbb{N}}$ of randomly distributed points, such that $\lim_{n\to\infty} |S_n| = \infty$. A point process is simple if the points are distinct. Let $N(\emptyset) = 0$ and denote the number of points in a measurable set $A \in \mathcal{B}(\mathcal{X})$ by

$$N(A) = \sum_{n\in\mathbb{N}} \mathbb{1}_{\{S_n \in A\}}.$$

Then $N : \Omega \to \mathbb{Z}_+^{\mathcal{B}(\mathcal{X})}$ is called a **counting process** for the simple point process S.

Remark 3. When $\mathcal{X} = \mathbb{R}_+$, one can order these points of S as an increasing sequence such that $S_1 < S_2 < \dots$, and denote number of points in half-open interval by

$$N(t) \triangleq N(0,t] = \sum_{n\in\mathbb{N}} \mathbb{1}_{(0,t]}(S_n) = \sum_{n\in\mathbb{N}} \mathbb{1}_{\{S_n \leqslant t\}}.$$

Remark 4. For a simple point process, we have $P(\{X_n = 0\}) = P(\{X_n \leqslant 0\}) = 0$.

Remark 5. General point processes in higher dimension don't have any inter-arrival time interpretation.

2 Renewal processes

We will consider **inter-arrival times** $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ to be a sequence of non-negative *iid* random variables with a common distribution F, such that $F(0) < 1$ and mean $\mu = \mathbb{E}X_1 = \int_{\mathbb{R}} x dF(x)$ is finite. We interpret X_n as the time between $(n-1)^{\text{st}}$ and the n^{th} renewal event.

Definition 2.1 (Renewal Instants). Let S_n denote the time of n^{th} **renewal instant** and assume $S_0 = 0$. Then, we have

$$S_n \triangleq \sum_{i=1}^{n} X_i, \quad n \in \mathbb{N}.$$

The random sequence $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is called sequence of renewal instants or renewal sequence.

Remark 6. The condition $F(0 < 1$ on inter-arrival times implies non-degenerate renewal process. If $F(0)$ is equal to 1 then it is a trivial process.

Definition 2.2 (Renewal process). The associated counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ that counts number of renewal until time t with *iid* general inter-renewal times is called a **renewal process**, written as

$$N(t) \triangleq \sup\{n \in \mathbb{Z}_+ : S_n \leqslant t\} = \sum_{n\in\mathbb{N}} \mathbb{1}_{\{S_n \leqslant t\}}.$$

Definition 2.3. A renewal process S is said to be **recurrent** if the inter-renewal time X_n is finite almost surely for every $n \in \mathbb{N}$, the process is called **transient** otherwise.

Definition 2.4. The process is said to be **periodic** with period d if the inter-renewal times $X : \Omega \to \mathcal{X}^{\mathbb{N}}$ take values in a discrete set $\mathcal{X} \subseteq \{nd : n \in \mathbb{Z}_+\}$ and $d = \gcd(\mathcal{X})$ is the largest such number. Otherwise, if there is no such $d > 0$, then the process is said to be **aperiodic**. If the inter-arrival times is a periodic random variable, then the associated distribution function F is called **lattice**.

Example 2.5 (Random walk). Random walk S on $\mathbb{R}^d$ with *iid* non-negative step-sizes $(X_n : n \in \mathbb{N})$ is a renewal process.

Example 2.6 (Markov chain). Let $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$ be a discrete time homogeneous Markov chain X with state space $\mathcal{X}$. For $X_0 = i \in \mathcal{X}, \tau_i^+(0) = 0$, let the recurrent times be defined inductively as

$$\tau_i(n)^+ = \inf\left\{k > \tau_i^+(n-1) : X_k = i\right\}. \tag{2}$$

It follows from the strong Markov property of the process X, that $\tau_i^+ : \Omega \to \mathbb{R}_+^{\mathbb{Z}_+}$ is a renewal sequence.

3 Delayed renewal process

Many times in practice, we have a *delayed start* to a renewal process. That is, the arrival process has independent inter-arrival times $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$, where the common distribution for X_n is F when $n \geqslant 2$, and the distribution of first inter-arrival time X_1 is G. Let the first renewal instant be $S_0 = 0$ and nth arrival instant be $S_n \triangleq \sum_{i=1}^{n} X_i$ for all $n \in \mathbb{N}$.

The associated counting process is called a **delayed renewal process** and denoted by $N_D : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$. The following inverse relationship continues to hold between counting and arrival process,

$$N_D(t) = \sup\{n \in \mathbb{N} : S_n \leqslant t\}. \tag{3}$$

Example 3.1 (Markov chain). Let $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$ be a discrete time homogeneous Markov chain X with state space V. For $X_0 = x \in \mathcal{X}$ and for $y \neq x$ let $\tau_y^+(0) = 0$, let the recurrent times be defined inductively as

$$\tau_y(k)^+ \triangleq \inf\left\{n > \tau_y^+(k-1) : X_n = y\right\}.$$

It follows from the strong Markov property of the process X, that $\tau_j^+ : \Omega \to \mathbb{N}^{\mathbb{Z}_+}$ is a delayed renewal sequence.