Lecture-07: Properties of Renewal Process

1 Properties of renewal processes

1.1 Finiteness of counting process

Lemma 1.1 (Finiteness). *For a renewal process with positive $\mathbb{E}X_n > 0$, the number of renewals $N(t)$ in the time duration $(0,t]$ is a.s. finite for all $t > 0$.*

Proof. We are interested in knowing how many renewals occur per unit time. From strong law of large numbers, we know that

$$P\left\{\lim_{n\in\mathbb{N}} \frac{S_n}{n} = \mu\right\} = 1. \tag{1}$$

Since $\mu > 0$, we must have S_n growing arbitrarily large as n increases. Thus, S_n can be finite for at most finitely many n. Indeed for any finite t, we have the the following set inclusion

$$\bigcap_{n\in\mathbb{N}} \{N(t) \geqslant n\} = \bigcap_{n\in\mathbb{N}} \{S_n \leqslant t\} \subseteq \bigcap_{n\in\mathbb{N}} \left\{\frac{S_n}{n} \leqslant \frac{t}{n}\right\} \subseteq \left\{\limsup_{n\in\mathbb{N}} \frac{S_n}{n} = 0\right\}. \tag{2}$$

Since, $\left\{\limsup_{n\in\mathbb{N}} \frac{S_n}{n} = 0\right\} \subset \left\{\lim_{n\in\mathbb{N}} \frac{S_n}{n} = \mu\right\}^c$, it follows that $P\{N(t) = \infty\} = 0$ for any finite t. The result follows and $N(t) = \max\{n \in \mathbb{N}_0 : S_n \leqslant t\}$. ☐

Corollary 1.2. *For a delayed renewal process with positive $\mathbb{E}X_n > 0$ for $n \geqslant 2$ and finite mean $\mathbb{E}X_1$, the number of renewals $N_D(t)$ in the time duration $(0,t]$ is a.s. finite for all $t > 0$.*

1.2 Distribution functions

Lemma 1.3. *The distribution function of nth renewal instant $S_n = \sum_{i=1}^n X_i$ is given by $P\{S_n \leqslant t\} = F_n(t)$ for all $t \in \mathbb{R}$, where F_n is n-fold convolution of the distribution function F for a single i.i.d. inter-arrival time X_i. The distribution function F_n is computed inductively as $F_n = F_{n-1} * F$, where $F_1 = F$.*

Corollary 1.4. *The distribution function of nth arrival instant S_n for delayed renewal process is $G * F_{n-1}$.*

Corollary 1.5. *The distribution function of counting process $N_D(t)$ for the delayed renewal process is*

$$P\{N_D(t) = n\} = P\{S_n \leqslant t\} - P\{S_{n+1} \leqslant t\} = (G * F_{n-1})(t) - (G * F_n)(t). \tag{3}$$

1.3 Renewal function

Mean of the counting process $N(t)$ is called the **renewal function** denoted by $m(t) = \mathbb{E}[N(t)]$.

Proposition 1.6. *Renewal function can be expressed in terms of distribution of renewal instants as*

$$m(t) = \sum_{n\in\mathbb{N}} F_n(t).$$

Proof. Using the inverse relationship between counting process and the arrival instants, we can write

$$m(t) = \mathbb{E}[N(t)] = \sum_{n\in\mathbb{N}} P\{N(t) \geqslant n\} = \sum_{n\in\mathbb{N}} P\{S_n \leqslant t\} = \sum_{n\in\mathbb{N}} F_n(t).$$

We can exchange integrals and summations since the integrand is positive using monotone convergence theorem. ☐

Corollary 1.7. *The renewal function for the delayed renewal process $N_D(t)$ is $m_D = G * (1 + m)$.*

Proof. We can write the renewal function for the delayed renewal process as

$$m_D(t) = \mathbb{E}N_D(t) = \sum_{n\in\mathbb{N}} (G * F_{n-1})(t) = G(t) + (G * m)(t). \tag{4}$$

□

Corollary 1.8. *We denote the Laplace transform for the inter-arrival time distribution F by $\mathcal{L}(F) = \tilde{F}$, then the Laplace transform of the renewal function m is given by*

$$\tilde{m}(s) = \frac{1}{1-\tilde{F}(s)}, \quad \Re\left\{\tilde{F}(s)\right\} < 1.$$

Lemma 1.9. *Let the Laplace transforms for the distributions of the first inter-arrival time and the subsequent inter-arrival times be denoted by $\tilde{G} = \mathcal{L}(G)$ and $\tilde{F} = \mathcal{L}(F)$ respectively, then the Laplace transform of the renewal function m_D for the delayed renewal process is*

$$\tilde{m}_D(s) = \frac{\tilde{G}(s)}{1-\tilde{F}(s)}, \quad \Re\left\{\tilde{F}(s)\right\} < 1. \tag{5}$$

Proposition 1.10. *For renewal process with $\mathbb{E}X_n > 0$, the renewal function is bounded for all finite times.*

Proof. Since we assumed that $P\{X_n = 0\} < 1$, it follow from continuity of probabilities that there exists $\alpha > 0$, such that $P\{X_n \geqslant \alpha\} = \beta > 0$. We can define bivariate random variables

$$\bar{X}_n = \alpha 1_{\{X_n \geqslant \alpha\}} \leqslant X_n. \tag{6}$$

Note that since X_i's are *iid*, so are $\bar{X}_i$'s. Each $\bar{X}_i$ takes values in $\{0,\alpha\}$ with probabilities $1-\beta$ and β respectively. Let $\bar{N}(t)$ denote the renewal process with inter-arrival times $\bar{X}_n$, with arrivals at integer multiples of α. Then for all sample paths, we have

$$N(t) = \sum_{n\in\mathbb{N}} 1_{\left\{\sum_{i=1}^n X_i \leqslant t\right\}} = \sum_{n\in\mathbb{N}} 1_{\left\{\sum_{i=1}^n \bar{X}_i \leqslant t\right\}} = \bar{N}(t). \tag{7}$$

Hence, it follows that $\mathbb{E}N(t) \leq \mathbb{E}\bar{N}(t)$, and we will show that $\mathbb{E}\bar{N}(t)$ is finite. We can write the joint distribution of number of arrivals at each arrival instant $l\alpha$, as

$$P\{\bar{N}(0) = n_1, \bar{N}(\alpha) = n_2\} = P\left(\bigcap_{i=1}^{n_1}\{X_i \leq \alpha\} \bigcap \{X_{n_1+1} \geq \alpha, X_{n_2+1} \geq \alpha\} \bigcap_{i=2}^{n_2}\{X_{n_1+i} < \alpha\}\right) = (1-\beta)^{n_1}\beta(1-\beta)^{n_2-1}\beta. \tag{8}$$

It follows that the number of arrivals is independent at each arrival instant $k\alpha$ and geometrically distributed with mean $1/\beta$ and $(1-\beta)/\beta$ for $k \in \mathbb{N}$ and $k = 0$ respectively. Thus, for all $t \geqslant 0$,

$$\mathbb{E}N(t) \leqslant \mathbb{E}\bar{N}(t) \leqslant \frac{\lceil \frac{t}{\alpha} \rceil}{\beta} \leqslant \frac{\frac{t}{\alpha}+1}{\beta} < \infty. \tag{9}$$

□

Corollary 1.11. *For delay renewal process with $\mathbb{E}X_n > 0$, the renewal function is bounded at all finite times.*

A Convolution of distribution functions

Definition A.1. For two distribution functions $F, G : \mathbb{R} \to [0,1]$ the convolution of F and G is a distribution function $F * G : \mathbb{R} \to [0,1]$ defined as

$$(F * G)(x) \triangleq \int_{y\in\mathbb{R}} F(x-y)dG(y), \; x \in \mathbb{R}.$$

Remark 1. Verify that $F * G$ is indeed a distribution function. That is, the function $(F * G)$ is

(a) right continuous, i.e. $\lim_{x_n \downarrow x}(F * G)(x_n)$ exists,

(b) non-decreasing, i.e. $(F * G)(z) \geqslant (F * G)(x)$ for all $z \geqslant x$,

(c) having left limit of zero and right limit of unity, i.e. $\lim_{x\to-\infty}(F*G)(x) = 0, \lim_{x\to\infty}(F*G)(x) = 1$.

Remark 2. Verify that convolution is a symmetric and bi-linear operator. That is, for any distribution functions (F,G) and $(F_i : i \in [n]), (G_j : j \in [m])$ and vectors $\alpha \in \mathbb{R}^n, \beta \in \mathbb{R}^m$, we have

$$F*G = G*F, \qquad \Big(\sum_{i\in[n]} \alpha_i F_i\Big) * \Big(\sum_{j\in[m]} \beta_j G_j\Big) = \sum_{i\in[n]}\sum_{j\in[m]} \alpha_i\beta_j (F_i * G_j).$$

Lemma A.2. *Let X and Y be two independent random variables defined on the probability space $(\Omega, \mathcal{F}, P)$ with distribution functions F and G respectively, then the distribution of $X+Y$ is given by $F*G$.*

Proof. The distribution function of sum $X+Y$ is given by $H : \mathbb{R} \to [0,1]$ where for any $z \in \mathbb{R}$,

$$H(z) = \mathbb{E}\mathbb{1}_{\{X+Y\leqslant z\}} = \mathbb{E}[\mathbb{E}[\mathbb{1}_{\{X+Y\leqslant z\}}|\sigma(Y)]] = \mathbb{E}[F(z-Y)] = \int_{y\in\mathbb{R}_+} F(z-y)dG(y).$$

□

Definition A.3. Let $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ be an independent random sequence defined on the probability space $(\Omega, \mathcal{F}, P)$ with distribution function F, then the distribution of $S_n \triangleq \sum_{i=1}^n X_i$ is given by $F_n \triangleq F_{n-1} * F$ for all $n \geqslant 2$ and $F_1 = F$.

B Laplace transform of distribution functions

Definition B.1. For a distribution function $F : \mathbb{R} \to [0,1]$ the Laplace transform $\mathcal{L}(F)$ is a map $\tilde{F} : \mathbb{C} \to \mathbb{C}$ defined

$$\tilde{F}(s) \triangleq \int_{y\in\mathbb{R}} e^{-sy} dF(y),$$

where s lies in the region such that $\left|\tilde{F}(s)\right| < \infty$.

Lemma B.2. *The Laplace transform of convolution of two distribution functions is product of Laplace transform of individual distribution functions.*

Proof. Let $F, G : \mathbb{R} \to [0,1]$ be two distribution functions such that $\mathcal{L}(F) = \tilde{F}$ and $\mathcal{L}(G) = \tilde{G}$, then

$$\mathcal{L}(F*G)(s) = \int_{x\in\mathbb{R}} e^{-sx} \int_{y\in\mathbb{R}} dF(x-y)dG(y) = \int_{y\in\mathbb{R}} e^{-sy} dG(y) \int_{x-y\in\mathbb{R}} e^{-s(x-y)} dF(x-y) = \tilde{F}(s)\tilde{G}(s).$$

□

Corollary B.3. *Let X and Y be two independent random variables defined on the probability space $(\Omega, \mathcal{F}, P)$ with Laplace transform of distribution functions $\tilde{F}$ and $\tilde{G}$ respectively, then the Laplace transform of the distribution of $X+Y$ is given by $\tilde{F}\tilde{G}$.*

Corollary B.4. *Let $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ be an independent random sequence defined on the probability space $(\Omega, \mathcal{F}, P)$ with the Laplace transform of the distribution function given by $\tilde{F}$, then the Laplace transform of the distribution of $S_n \triangleq \sum_{i=1}^n X_i$ is given by $\mathcal{L}(F_n) = (\tilde{F})^n$.*