Lecture-08: Limit Theorems

1 Growth of renewal counting processes

Lemma 1.1. *Let $N(\infty) \triangleq \lim_{t\to\infty} N(t)$. For finite mean renewal processes, $P\{N(\infty) = \infty\} = 1$.*

Proof. It suffices to show $P\{N(\infty) < \infty\} = 0$. Since $\mathbb{E}[X_n] < \infty$, we have $P\{X_n = \infty\} = 0$ and

$$P\{N(\infty) < \infty\} = P\bigcup_{n\in\mathbb{N}}\{N(\infty) < n\} = P\bigcup_{n\in\mathbb{N}}\{S_n = \infty\} = P\bigcup_{n\in\mathbb{N}}\{X_n = \infty\} \leqslant \sum_{n\in\mathbb{N}} P\{X_n = \infty\} = 0. \quad (1)$$

□

Corollary 1.2. *For delayed renewal processes with finite mean of first renewal instant and subsequent inter-renewal times, $P\{\lim_{t\to\infty} N_D(t) = \infty\} = 1$.*

We observed that the number of renewals $N(t)$ increases to infinity with the length of the duration t. We will show that the growth of $N(t)$ is asymptotically linear with time t, and we will find this coefficient of linear growth of $N(t)$ with time t.

1.1 Strong law for renewal processes

Theorem 1.3 (Strong law). *For a renewal counting process with inter-arrival times having a finite mean, we have*

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu} \text{ almost surely}. \quad (2)$$

Proof. Note that $S_{N(t)}$ represents the time of last renewal before t, and $S_{N(t)+1}$ represents the time of first renewal after time t. Clearly, we have $S_{N(t)} \leqslant t < S_{N(t)+1}$. Dividing by $N(t)$, we get

$$\frac{S_{N(t)}}{N(t)} \leqslant \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)}. \quad (3)$$

Since $N(t)$ increases monotonically to infinity as t grows large, we can apply strong law of large numbers to the sum $S_{N(t)} = \sum_{i=1}^{N(t)} X_i$, to get $\lim_{t\to\infty} \frac{S_{N(t)}}{N(t)} = \mu$ almost surely. Hence the result follows. □

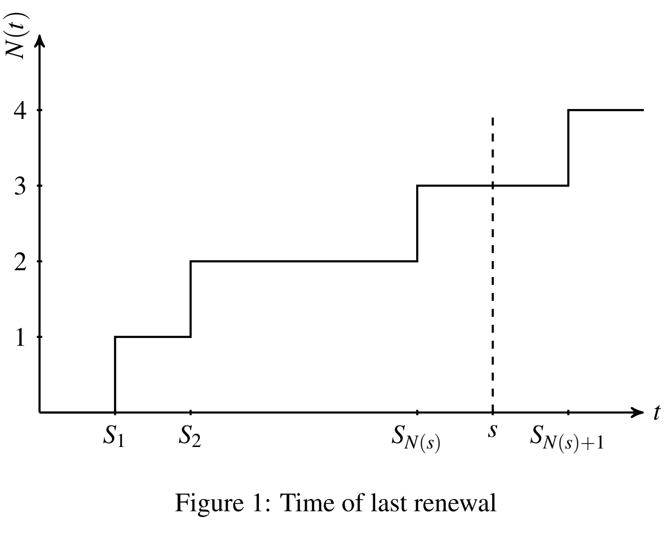

Figure 1: Time of last renewal

Corollary 1.4. *For a delayed renewal process with finite inter-arrival durations,* $\lim_{t\to\infty} \frac{N_D(t)}{t} = \frac{1}{\mu_F}$.

Example 1.5. Suppose, you are in a casino with infinitely many games. Every game has a probability of win X, *iid* uniformly distributed between $(0,1)$. One can continue to play a game or switch to another one. We are interested in a strategy that maximizes the long-run proportion of wins. Let $N(n)$ denote the number of losses in n plays. Then the fraction of wins $P_W(n)$ is given by

$$P_W(n) = \frac{n - N(n)}{n}.$$

We pick a strategy where any game is selected to play, and continue to be played till the first loss. Note that, time till first loss is geometrically distributed with mean $\frac{1}{1-X}$. We shall show that this fraction approaches unity as $n \to \infty$. By the previous proposition, we have:

$$\begin{aligned}\lim_{n\to\infty} \frac{N(n)}{n} &= \frac{1}{\mathbb{E}[\text{Time till first loss}]} \\ &= \frac{1}{\mathbb{E}\left[\frac{1}{1-X}\right]} = \frac{1}{\infty} = 0\end{aligned}$$

Hence Renewal theorems can be used to compute these long term averages. We'll have many such theorems in the following sections.

1.2 Elementary renewal theorem

Basic renewal theorem implies $\frac{N(t)}{t}$ converges to $\frac{1}{\mu}$ almost surely. We are next interested in convergence of the ratio $\frac{m(t)}{t}$. Note that this is not obvious, since almost sure convergence doesn't imply convergence in mean. To illustrate this, we have the following example.

Example 1.6. Let X_n be a Bernoulli random variable with $P\{X_n = 1\} = 1/n$. Let $Y_n = nX_n$. Then, $P\{Y_n = 0\} = 1 - 1/n$. That is $Y_n \to 0$ a.s. However, $\mathbb{E}[Y_n] = 1$ for all $n \in \mathbb{N}$. So $\mathbb{E}[Y_n] \to 1$.

Even though, basic renewal theorem does **NOT** imply it, we still have $\frac{m(t)}{t}$ converging to $\frac{1}{\mu}$. We first need this technical Lemma.

Proposition 1.7 (Wald's Lemma for Renewal Process). *Let* $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ *be iid inter-arrival times of a renewal process* $N(t)$ *with* $\mu = \mathbb{E}[X_1] < \infty$, *and let* $m(t) = \mathbb{E}[N(t)]$ *be its renewal function. Then,* $N(t)+1$ *is a stopping time and*

$$\mathbb{E}\left[\sum_{i=1}^{N(t)+1} X_i\right] = \mu(1 + m(t)).$$

Proof. We observe that for any $n \in \mathbb{N}$, the event $\{N(t)+1 = n\}$ belongs to $\sigma(X_1, \dots, X_n)$, since

$$\{N(t)+1 = n\} = \{S_{n-1} \leqslant t < S_n\} = \left\{\sum_{i=1}^{n-1} X_i \leqslant t < \sum_{i=1}^{n-1} X_i + X_n\right\} \in \sigma(X_1, \dots, X_n).$$

Thus $N(t)+1$ is a stopping time with respect to the random sequence X, and the result follows from Wald's Lemma. ☐

Theorem 1.8 (Elementary renewal theorem). *For a renewal process with finite mean inter-arrival times, the renewal function satisfies*

$$\lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu}. \tag{4}$$

Proof. By the assumption, we have mean $\mu < \infty$. Further, we know that $S_{N(t)+1} > t$. Taking expectations on both sides and using Proposition 1.7, we have $\mu(m(t)+1) > t$. Dividing both sides by μt and taking $\liminf$ on both sides, we get

$$\liminf_{t\to\infty} \frac{m(t)}{t} \geqslant \frac{1}{\mu}. \tag{5}$$

We employ a truncated random variable argument to show the reverse inequality. We define truncated inter-arrival times $(\bar{X}_n = \min(X_n, M) : n \in \mathbb{N})$ with mean denoted by μ_M. These modified inter-arrival times are *iid* and hence we can define the corresponding renewal process $(\bar{S}_n = \sum_{i=1}^n \bar{X}_i : n \in \mathbb{N})$ and the associated counting process $\bar{N}(t) = \sum_{n\in\mathbb{N}} 1_{\left\{\bar{S}_n \leqslant t\right\}}$. Note that since $S_n \geqslant \bar{S}_n$, the number of arrivals would be higher for renewal process $\bar{N}(t)$ with truncated random variables. That is,

$$N(t) \leqslant \bar{N}(t). \tag{6}$$

Further, due to truncation of inter-arrival time, next renewal happens within M units of time, that is $\bar{S}_{\bar{N}(t)+1} \leq t+M$. Taking expectations on both sides in the above equation, using Proposition 1.7, dividing both sides by $t\mu_M$, and taking $\limsup$ on both sides, we obtain

$$\limsup_{t\to\infty} \frac{\bar{m}(t)}{t} \leqslant \frac{1}{\mu_M}.$$

Recognizing that $\lim_{M\to\infty} \mu_M = \mu$, the result follows from taking expectations on both sides of (6), and the lower bound on $\liminf$ on the ratio $m(t)/t$. ☐

Corollary 1.9. *For a delayed renewal process with finite inter-arrival durations, we have* $\lim_{t\to\infty} \frac{m_D(t)}{t} = \frac{1}{\mu_F}$.

Example 1.10 (Markov chain). Consider a positive recurrent discrete time Markov chain $(X_n \in V : n \in \mathbb{N})$ taking values in a discrete set V. Let the initial state be $X_0 = i \in V$ and $\tau_j^+(0) = 0$ for $j \neq i \in V$, then we can inductively define the *n*th recurrent time to state j as a stopping time

$$\tau_j^+(n) = \inf\left\{k > \tau_j^+(n-1) : X_k = j\right\}. \tag{7}$$

Since any discrete time Marko chain satisfies the strong Markov property, it follows that $(\tau_j^+(n) : n \in \mathbb{N})$ form a delayed renewal process with the first arrival distribution $P_i\left\{\tau_j^+ = k\right\} = f_{ij}^{(k)}$, and the common distribution of the inter-arrival duration $X_n, n \geqslant 2$ in terms of first return probability as

$$P_j\left\{\tau_j^+ = k\right\} = f_{jj}^{(k)},\ k \in \mathbb{N}. \tag{8}$$

We denote the associated counting process by $(N_j(n) : n \in \mathbb{N})$, where $N_j(n) = \sum_{i\in\mathbb{N}} 1_{\{S_i \leqslant n\}} = \sum_{k=1}^n 1_{\{X_k = j\}}$ denotes the number of transitions to state j up to time n. Let $\mu_{jj} = \mathbb{E}_j \tau_j^+$ be the finite mean inter-arrival time for the renewal process, also the mean recurrence time to state j. From the strong law for delayed renewal processes it follows that

$$P_j\left\{\lim_{n\in\mathbb{N}} \frac{N_j(n)}{n} = \frac{1}{\mu_{jj}}\right\} = 1. \tag{9}$$

Since $N_j(n)$ is number of visits to state j in first n time steps, we have $\mathbb{E}_i N_j(n) = \sum_{k=1}^n P_i(X_k = j) = \sum_{k=1}^n p_{ij}^{(k)}$ From the basic renewal theorem for delayed renewal process it follows that

$$\lim_{n\in\mathbb{N}} \frac{\sum_{k=1}^n p_{ij}^{(k)}}{n} = \lim_{n\in\mathbb{N}} \frac{\mathbb{E}_i[N_j(n)]}{n} = \frac{1}{\mu_{jj}}. \tag{10}$$

1.3 Central limit theorem for renewal processes

Theorem 1.11. *For a renewal process with inter-arrival times having finite mean* μ *and finite variance* σ^2*, the associated counting process converges to a normal random variable in distribution. Specifically,*

$$\lim_{t\to\infty} P\left\{\frac{N(t)-\frac{t}{\mu}}{\sigma\sqrt{\frac{t}{\mu^3}}} < y\right\} = \int_{-\infty}^{y} e^{-\frac{x^2}{2}}\,dx.$$

Proof. Take $u = \frac{t}{\mu} + y\sigma\sqrt{\frac{t}{\mu^3}}$. We shall treat u as an integer and proceed, the proof for general u is an exercise. Recall that $\{N(t) < u\} = \{S_u > t\}$. By equating probability measures on both sides, we get

$$P\{N(t) < u\} = P\left\{\frac{S_u - u\mu}{\sigma\sqrt{u}} > \frac{t-u\mu}{\sigma\sqrt{u}}\right\} = P\left\{\frac{S_u - u\mu}{\sigma\sqrt{u}} > -y\left(1+\frac{y\sigma}{\sqrt{tu}}\right)^{-1/2}\right\}.$$

By central limit theorem, $\frac{S_u-u\mu}{\sigma\sqrt{u}}$ converges to a normal random variable with zero mean and unit variance as t grows. We also observe that

$$\lim_{t\to\infty} -y\left(1+\frac{y\sigma}{\sqrt{tu}}\right)^{-1/2} = -y.$$

These results combine with the symmetry of normal random variable to give us the result. □