Lecture-09: Regenerative Processes

1 Regenerative processes

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ be a renewal sequence, with the associated inter-renewal sequence $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ and the counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$. That is, we have $S_n \triangleq \sum_{i=1}^n X_i$ for each $n \in \mathbb{N}$ and $N(t) = \sum_{n \in \mathbb{N}} \mathbb{1}_{\{S_n \leqslant t\}}$ for each $t \in \mathbb{R}_+$.

Consider a stochastic process $Z : \Omega \to \mathbb{R}^{\mathbb{R}_+}$ defined over the same probability space, where the nth segment of the joint process $(N, Z) : \Omega \to (\mathbb{Z}_+ \times \mathbb{R})^{\mathbb{R}_+}$ is defined as the sample path in the nth inter-renewal duration, written

$$\zeta_n \triangleq (X_n, (Z_t : t \in [S_{n-1}, S_n)), \qquad n \in \mathbb{N}.$$

Definition 1.1. The process Z is regenerative over the renewal sequence S, if its segments $(\zeta_n : n \in \mathbb{N})$ are *i.i.d.* The process Z is delayed regenerative, if S is a delayed renewal sequence and the segments of the joint process are independent with $(\zeta_n : n \geqslant 2)$ being identically distributed.

Remark 1. Let $\mathcal{F}_t = \sigma(N(u), Z(u), u \leqslant t)$ be the history until time $t \in \mathbb{R}_+$. The renewal sequence S is the **regeneration times** for the process Z, and the process Z possesses the **regenerative property** that the process $(Z_{S_{n-1}+t} : t \geqslant 0)$ is independent of history $\mathcal{F}_{S_{n-1}}$.

Remark 2. The definition says that probability law is independent of the past and shift invariant at renewal times. That is after each renewal instant, the process becomes an independent probabilistic replica of the process starting from zero.

Remark 3. If the stochastic process Z is bounded, then for any Borel measurable function $f : \mathbb{R} \to \mathbb{R}$, we have

$$\mathbb{E}[f(Z_{S_{n-1}+t}) | \mathcal{F}_{S_{n-1}}] = \mathbb{E} f(Z_t).$$

Example 1.2 (Age process). Let $N : \Omega \to \mathbb{R}_+^{\mathbb{R}_+}$ be the renewal counting process for the renewal sequence $(S_n : n \in \mathbb{N})$, then the age at time t is defined as $A(t) = t - S_{N(t)}$. Then the age process $A : \Omega \to \mathbb{R}_+^{\mathbb{R}_+}$ is regenerative. To see this, we observe that the sample path of age in nth renewal interval is given by

$$A(S_{n-1} + t) = t, \qquad t \in [0, X_n).$$

Since the segments $(X_n, (t : t \in [0, X_n)))$ are *i.i.d.*, the result follows.

Example 1.3 (Markov chains). For a discrete time irreducible and positive recurrent homogeneous Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{N}}$ on finite state space $\mathcal{X} \subset \mathbb{R}$, we can inductively define the recurrent times for state $y \in \mathcal{X}$ as $\tau_y^+(0) = 0$, and

$$\tau_y^+(n) = \inf\left\{k > \tau_y^+(n-1) : X_k = y\right\}.$$

From the strong Markov property of Markov chain X, it follows that $\tau_y^+ : \Omega \to \mathbb{N}^{\mathbb{N}}$ is a delayed renewal sequence. For all $n \in \mathbb{N}$, we define the nth excursion time to the state y as $I_n \triangleq \left\{\tau_y^+(n-1), \dots, \tau_y^+(n) - 1\right\}$ and length of this excursion as $T_y(n) \triangleq \tau_y^+(n) - \tau_y^+(n-1)$.

We show that the Markov process X is regenerative over renewal sequence τ_y^+. Let the nth excursion time to the state y be denoted by $I_n = \left\{\tau_y^+(n-1), \dots, \tau_y^+(n) - 1\right\}$, then we can write the nth segment as $\zeta_n = (T_y(n), (X_k, k \in I_n))$.

Independence of the segments follows from the strong Markov property. Further, in the nth segment of the joint process, we can write the joint distribution for $(T_y(n), X_{\tau_y^+(n-1)+k})$ for $k < T_y(n)$ and $x \neq y$ as

$$P\left\{T_y(n) = m, X_{\tau_y^+(n-1)+k} = x, k \in [m-1]\right\} = P_y\left\{\tau_y^+(1) > k, X_k = x\right\} P_x\left\{\tau_y^+(1) = m - k\right\}.$$

The equality follows f rom the strong Markov property and the homogeneity of process X.

2 Renewal equation

Let $Z : \Omega \to \mathbb{R}^{\mathbb{R}_+}$ be a regenerative process over renewal sequence $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ defined on the probability space $(\Omega, \mathcal{F}, P)$, and F be the distribution of inter-renewal times. The counting process associated with the renewal sequence S is denoted by N, and we define the history of the joint process Z, N until time t by $\mathcal{F}_t$. For any Borel measurable set $A \in \mathcal{B}(\mathbb{R})$ and time $t \geqslant 0$, we are interested in computing time dependent marginal probability $f(t) = P\{Z_t \in A\}$. We can write the probability of the event $\{Z_t \in A\}$ by partitioning it into disjoint events as

$$P\{Z_t \in A\} = P\{Z_t \in A, S_1 > t\} + P\{Z_t \in A, S_1 \leqslant t\}.$$

We define the kernel function $K(t) = P\{S_1 > t, Z_t \in A\}$ which are typically easy to compute for any regenerative process. By the regeneration property applied at renewal instant S_1, we have

$$\mathbb{E}[\mathbb{E}[\mathbb{1}_{\{Z_t \in A, S_1 \leqslant t\}} | \sigma(S_1)] = \mathbb{E}[\mathbb{E}[\mathbb{1}_{\{Z_t \in A, S_1 \leqslant t\}} | \mathcal{F}_{S_1}] | \sigma(S_1)] = \mathbb{1}_{\{S_1 \leqslant t\}} \mathbb{E}[\mathbb{1}_{\{Z_{t-S_1} \in A\}} | \sigma(S_1)] = f(t - S_1) \mathbb{1}_{\{S_1 \leqslant t\}}.$$

Hence, we have the following fixed point **renewal equation** for f

$$f(t) = K(t) + \int_0^t dF(s) f(t-s) = K + F * f.$$

We assume that the distribution function F and the kernel K are known, and we wish to find f, and characterize its asymptotic behavior.

Example 2.1 (Age and Excess time processes). For a renewal sequence $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ with associated counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$, we can define the age process $A : \Omega \to \mathbb{R}_+^{\mathbb{R}_+}$ where the age $A(t)$ at time t is the time since last renewal, i.e.

$$A(t) \triangleq t - S_{N(t)}, \; t \in \mathbb{R}_+.$$

Similarly, we can define the excess time process $Y : \Omega \to \mathbb{R}_+^{\mathbb{R}_+}$ where the excess time $Y(t)$ at time t is the time until next renewal, i.e.

$$Y(t) \triangleq S_{N(t)+1} - t, \; t \in \mathbb{R}_+.$$

Since the age process is regenerative for the associated renewal sequence, we can write the renewal equation for its distribution function as

$$P\{A(t) \geqslant x\} = P\{A(t) \geqslant x, S_1 > t\} + \int_0^t dF(y) P\{A(t-y) \geqslant x\}.$$

Theorem 2.2. *The renewal equation has a unique solution $f = (1+m) * K$, where $m(t) = \sum_{n \in \mathbb{N}} F_n(t)$ is the renewal function associated with the inter-renewal time distribution F.*

Proof. It follows from the renewal equation that $F * (1+m) * K = \sum_{n \in \mathbb{N}} F_n * K = m * K$. Hence, it is clear that $(1+m) * K$ is a solution to the renewal equation. For uniqueness, let f be another solution, then $h = f - K - m * K$ satisfies $h = F * h$, and hence $h = F_n * h$ for all $n \in \mathbb{N}$. From finiteness of $m(t)$, it follows that $F_n(t) \to 0$ as n grows. Hence, $\lim_{n \in \mathbb{N}} (F_n * h)(t) = 0$ for each t. □

Proposition 2.3. *Let Z be a regenerative process with state space $\mathcal{X} \subset \mathbb{R}$, over a renewal sequence S with renewal function m. For a Borel measurable set $A \in \mathcal{B}(\mathbb{R})$ and the kernel function $K(t) = P\{Z_t \in A, S_1 > t\}$, we can write for any $t \geqslant 0$*

$$P\{Z_t \in A\} = K(t) + \int_0^t dm(s) K(t-s). \tag{1}$$

Example 2.4 (Age process). Since the age process is regenerative for the associated renewal sequence, we can write the kernel function $K(t)$ in the renewal equation for its distribution function in terms of the complementary distribution function $\bar{F}$ of the inter-arrival times, as $K(t) = P\{A(t) \geqslant x, S_1 > t\} = 1_{\{t \geqslant x\}} \bar{F}(t)$. From the solution of renewal equation it follows that

$$P\{A(t) \geqslant x\} = 1_{\{t \geqslant x\}} \bar{F}(t) + \int_0^t dm(y) 1_{\{t-y \geqslant x\}} \bar{F}(t-y). \tag{2}$$