Lecture-10: Key Lemma and Blackwell Theorem

1 Key Lemma

Theorem 1.1 (Key Lemma). *Let $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ be a renewal process with i.i.d.inter-renewal times $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ having common distribution function F, associated counting process $N(t)$, and the renewal function $m(t)$. Then,*

$$P\left\{S_{N(t)} \leqslant s\right\} = \bar{F}(t) + \int_0^s \bar{F}(t-y)dm(y), \qquad t \geq s \geq 0.$$

Proof. We can see that event of time of last renewal prior to t being smaller than another time s can be partitioned into disjoint events corresponding to number of renewals till time t. Each of these disjoint events is equivalent to occurrence of nth renewal before time s and $(n+1)$th renewal past time t. That is,

$$\left\{S_{N(t)} \leqslant s\right\} = \bigcup_{n \in \mathbb{Z}_+} \left\{S_{N(t)} \leqslant s, N(t) = n\right\} = \bigcup_{n \in \mathbb{Z}_+} \left\{S_n \leqslant s, S_{n+1} > t\right\}.$$

Recognizing that $S_0 = 0, S_1 = X_1$, and that $S_{n+1} = S_n + X_{n+1}$, we can write

$$P\left\{S_{N(t)} \leqslant s\right\} = P\left\{X_1 > t\right\} + \sum_{n \in \mathbb{N}} \mathbb{E}[\mathbb{1}_{\{S_n \leqslant s\}} \mathbb{E}[\mathbb{1}_{\{X_{n+1} > t - S_n\}} | \sigma(S_n)]].$$

We recall F_n, n-fold convolution of F, is the distribution function of S_n. Taking expectation of $\bar{F}(t-S_n)\mathbb{1}_{\{S_n \leqslant s\}}$, we get

$$P\left\{S_{N(t)} \leqslant s\right\} = \bar{F}(t) + \sum_{n \in \mathbb{N}} \int_{y=0}^s \bar{F}(t-y)dF_n(y).$$

Using monotone convergence theorem to interchange integral and summation, and noticing that $m(y) = \sum_{n \in \mathbb{N}} F_n(y)$, the result follows. ☐

Remark 1. A simple proof of key lemma follows from the fact that $A(t) = t - S_{N(t)}$ is a regenerative process. Then $f(t) = P\{A(t) \geqslant x\}$ for $x \in [0,t]$ and $f(t) = 0$ for $x \geqslant t$. Hence, we can write the corresponding kernel function is

$$K(t) = P\{S_1 > t, A(t) \geqslant x\} = P\{S_1 > t, t \geqslant x\} = \bar{F}(t)1_{\{t \geqslant x\}}.$$

Therefore, it follows that

$$P\left\{S_{N(t)} \leqslant s\right\} = P\left\{A(t) \geqslant t-s\right\} = \bar{F}(t)\mathbb{1}_{\{t \geqslant t-s\}} + \int_0^t dm(y)\bar{F}(t-y)\mathbb{1}_{\{t-y \geqslant t-s\}}.$$

Remark 2. Key lemma tells us that distribution of $S_{N(t)}$ has probability mass at 0 and density between $(0,t]$, that is,

$$\Pr\{S_{N(t)} = 0\} = \bar{F}(t), \qquad dF_{S_{N(t)}}(y) = \bar{F}(t-y)dm(y) \quad 0 < y \leqslant t.$$

Remark 3. Density of $S_{N(t)}$ has interpretation of renewal taking place in the infinitesimal neighborhood of y, and next inter-arrival after time $t-y$. To see this, we notice

$$dm(y) = \sum_{n \in \mathbb{N}} dF_n(y) = \sum_{n \in \mathbb{N}} P\left\{S_n \in (y, y+dy)\right\} = \sum_{n \in \mathbb{N}} P\left\{n\text{th renewal occurs in } (y, y+dy)\right\}.$$

Combining interpretation of density of inter-arrival time $dF(t)$, we get

$$dF_{S_{N(t)}}(y) = \Pr\{\text{renewal occurs in } (y, y+dy) \text{ and next arrival after } t-y\}. \tag{1}$$

2 Delayed Regenerative Process

Theorem 2.1. *Let Z be a delayed regenerative process with the associated delayed renewal sequence S, the renewal function m_D, the first arrival distribution G, and the common inter-arrival duration distribution F. For a Borel measurable set $A \in \mathcal{B}(\mathbb{R})$, we define the kernel functions $K_1(t) \triangleq P\{Z_t \in A, S_1 > t\}, K_2(t) \triangleq P\{Z_{S_1+t} \in A, t \in [0, X_2)\}$, then we have*

$$P\{Z_t \in A\} = K_1(t) + \int_0^t dm_D(y) K_2(t-y). \tag{2}$$

Proof. For a Borel measurable set $A \in \mathcal{B}(\mathbb{R})$, we can write the probability of the delayed regenerative process taking values in this set as disjoint sum of probability of disjoint partitions of this event as

$$P\{Z_t \in A\} = P\{Z_t \in A, S_1 > t\} + \sum_{n \in \mathbb{N}} P\{Z_t \in A, N(t) = n\}.$$

The nth segment of the joint process $(N_D(t), Z(t))$ is $\zeta_n = (X_n, (Z(S_{n-1}+t) : t \in [0, X_n)))$. From the regenerative property, we know that the segments $(\zeta_n : n \in \mathbb{N})$ are independent, where $(\zeta_n : n \geqslant 2)$ are identically distributed. In particular, we can write

$$\mathbb{E}[\mathbb{E}\mathbb{1}_{\{Z_t \in A, S_n \leqslant t < S_{n+1}\}} | \mathcal{F}_{S_n}] | \sigma(S_n)] = \mathbb{1}_{\{S_n \leqslant t\}} \mathbb{E}[\mathbb{1}_{\{Z_{S_1+t-S_n} \in A, t-S_n \in [0, X_2)\}} \mid \sigma(S_n)] = \mathbb{1}_{\{S_n \leqslant t\}} K_2(t - S_n).$$

The result follows from the fact that $P\{Z_t \in A, N(t) = n\} = \mathbb{E}[\mathbb{1}_{\{Z_t \in A, S_n \leqslant t < S_{n+1}\}} | \sigma(S_n)]$. ☐

Example 2.2 (Age process). Age process $(A(t) = t - S_{N(t)} : t \geqslant 0)$ for a delayed renewal process $(S_n : n \in \mathbb{N})$ is a delayed regenerative process, since the nth segment is given by $\zeta_n = (X_n, (A(S_{n-1}+t) = t : t \geqslant [0, X_n)))$. For the measurable set $B = [x, \infty)$, then we can compute the kernel functions

$$K_1(t) = P\{A(t) \geqslant x, S_1 > t\} = 1_{\{t \geqslant x\}} \bar{G}(t), \qquad K_2(t) = P\{A(S_1+t) \geqslant x, t \in [0, X_2)\} = 1_{\{t \geqslant x\}} \bar{F}(t).$$

Therefore, we can write the distribution of last renewal time for the delayed renewal process as

$$P\left\{S_{N(t)} \leqslant x\right\} = P\{A(t) \geqslant t - x\} = \mathbb{1}_{\{x \geqslant 0\}} \bar{G}(t) + \int_0^t dm_D(y) \mathbb{1}_{\{t-y \geqslant t-x\}} \bar{F}(t-y).$$

Corollary 2.3 (Delayed Key Lemma). *Let $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ be a delayed renewal process with independet inter-renewal times $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ with first renewal time having distribution G and common distribution F for inter-renewal times $(X_n, n \geqslant 2)$, associated counting process $N_D(t)$, and the renewal function $m_D(t)$. Then,*

$$P\left\{S_{N(t)} \leqslant s\right\} = \bar{G}(t) + \int_0^s \bar{F}(t-y) dm_D(y), \qquad t \geq s \geq 0.$$

3 Blackwell Theorem

Lemma 3.1. *Let F be the inter-renewal distribution such that* $\inf\{x : F(x) = 1\} = \infty$*, then for any $b > 0$*

$$\sup_t \{m(t) - m(t-b)\} < \infty.$$

Proof. Recall that $m = \sum_{n \in \mathbb{N}} F_n$ and hence $m * F = m - F$. This implies that $m * (1 - F) = F$. Since the function $1 - F$ is monotonically non-increasing, $\inf_{s \in [0,b]} \bar{F}(s) = \bar{F}(b)$. Therefore,

$$1 \geqslant F(t) = \int_0^t dm(s) \bar{F}(t-s) \geqslant \int_{t-b}^t dm(s) \bar{F}(t-s) \geqslant [m(t) - m(t-b)] \bar{F}(b),$$

where b is chosen so that $F(b) < 1$. Hence, the result follows. ☐

Theorem 3.2 (Blackwell's Theorem). *Let the inter-renewal times have distribution F, mean μ, and the associated renewal function $m(t)$, such that* $\inf\{x : F(x) = 1\} = \infty$*. If F is not lattice, then for all $a \geq 0$*

$$\lim_{t \to \infty} m(t+a) - m(t) = \frac{a}{\mu}.$$

If F is lattice with period d, then

$$\lim_{n \to \infty} m((n+1)d) - m(nd) = \frac{d}{\mu}.$$

Proof. We will not prove that the following limit exists for non-lattice F,

$$g(a) = \lim_{t\to\infty}[m(t+a) - m(t)] \tag{3}$$

However, we show that if this limit does exist, it is equal to a/μ as a consequence of elementary renewal theorem. To this end, note that

$$m(t+a+b) - m(t) = m(t+a+b) - m(t+a) + m(t+a) - m(t).$$

Taking limits on both sides of the above equation, we conclude that $g(a+b) = g(a) + g(b)$. The only increasing solution of such a g is

$$g(a) = ca, \forall a > 0,$$

for some positive constant c. To show $c = \frac{1}{\mu}$, define a sequence $\{x_n, n \in \mathbb{N}\}$ in terms of $m(t)$ as

$$x_n = m(n) - m(n-1),\ n \in \mathbb{N}.$$

Note that $\sum_{i=1}^{n} x_i = m(n)$ and $\lim_{n\in\mathbb{N}} x_n = g(1) = c$, hence we have

$$\lim_{n\in\mathbb{N}} \frac{\sum_{i=1}^{n} x_i}{n} = \lim_{n\in\mathbb{N}} \frac{m(n)}{n} \overset{(a)}{=} c,$$

where (a) follows from the fact that if a sequence $\{x_i\}$ converges to c, then the running average sequence $a_n = \frac{1}{n}\sum_{i=1}^{n} x_i$ also converges to c, as $n \to \infty$. Therefore, we can conclude $c = 1/\mu$ by elementary renewal theorem.

When F is lattice with period d, the limit in (3) doesn't exist. (See the following example). However, the theorem is true for lattice again by elementary renewal theorem. Indeed, since $\frac{m(nd)}{n} \to \frac{1}{\mu}$, we can define $x_n = m(nd) - m((n-1)d)$ and observe that $\sum_{i=1}^{n} x_n = m(nd)$ and $\frac{1}{n}\sum_{i=1}^{n} x_n$ converges to $\frac{d}{\mu}$ by elementary renewal theorem. □

Example 3.3. For a trivial lattice example where the $\lim_{t\to\infty} m(t+a) - m(t)$ does not exist, consider a renewal process with $\Pr\{X_n = 1\} = 1$, that is, there is a renewal at every positive integer time instant with probability 1. Then F is lattice with $d = 1$. Now, for $a = 0.5$, and $t_n = n + (-1)^n 0.5$, we see that $\lim_{t_n\to\infty} m(t_n + a) - m(t_n)$ does not exist, and hence $\lim_{t\to\infty} m(t+a) - m(t)$ does not exist.

Remark 4. In the lattice case, if the inter arrivals are strictly positive, that is, there can be no more than one renewal at each nd, then we have that

$$\lim_{n\to\infty} P[\text{renewal at nd}] = \frac{d}{\mu}. \tag{4}$$

Corollary 3.4 (Delayed Blackwell's Theorem). *Consider a delayed renewal process with independent inter-renewal times, with the distribution of first renewal being G with mean μ_G, and distribution of inter-renewal times for $n \geqslant 2$ being F with mean μ_F and the property* $\inf\{x : F(x) = 1\} = \infty$. *Let the associated renewal function be $m_D(t)$ and F is not lattice, then for all $a \geqslant 0$*

$$\lim_{t\to\infty} m_D(t+a) - m_D(t) = \frac{a}{\mu_F}.$$

If F and G are lattice with period d, then

$$\lim_{n\to\infty} m_D((n+1)d) - m_D(nd) = \frac{d}{\mu_F}.$$