Lecture-11: Key Renewal Theorem

1 Key Renewal Theorem

**Theorem 1.1 (Key renewal theorem).** Consider a recurrent renewal process \( S : \Omega \to \mathbb{R}_+^\mathbb{N} \) with renewal function \( m(t) \), and the common mean and the distribution of i.i.d. inter-renewal times being denoted by \( \mu \) and \( F \) respectively. For any directly Riemann integrable function \( z \in \mathcal{D} \), we have

\[
\lim_{t \to \infty} \int_0^t z(t-x)dm(x) = \begin{cases} 
\frac{1}{\mu} \int_0^\infty z(t)dt, & F \text{ is non-lattice}, \\
\frac{1}{\mu} \sum_{k \in \mathbb{Z}_+} z(t+kd), & F \text{ is lattice with period } d, \quad t = nd.
\end{cases}
\]

**Proposition 1.2 (Equivalence).** Blackwell’s theorem and key renewal theorem are equivalent.

**Proof.** Let’s assume key renewal theorem is true. We select \( z : \mathbb{R}_+ \to \mathbb{R}_+ \) as a simple function with value unity on interval \([0,a]\) for \( a > 0 \) and zero elsewhere. That is, \( z(t) = 1_{[0,a]}(t) \) for any \( t \in \mathbb{R}_+ \). From Proposition A.3 it follows that \( z \) is directly Riemann integrable. Therefore, by Key Renewal Theorem, we have

\[
\lim_{t \to \infty} [m(t) - m(t-a)] = \frac{a}{\mu}.
\]

We defer the formal proof of converse for a later stage. We observe that, from Blackwell theorem, it follows

\[
\lim_{t \to \infty} \frac{dm(t)}{dt} \overset{(a)}{=} \lim_{a \to 0} \frac{m(t+a) - m(t)}{a} = \frac{1}{\mu},
\]

where in \((a)\) we can exchange the order of limits under certain regularity conditions.

**Remark 1.** Key renewal theorem is very useful in computing the limiting value of some function \( g(t) \), probability or expectation of an event at an arbitrary time \( t \), for a renewal process. This value is computed by conditioning on the time of last renewal prior to time \( t \).

**Corollary 1.3 (Delayed key renewal theorem).** Consider an aperiodic and recurrent delayed renewal process \( S : \Omega \to \mathbb{R}_+^\mathbb{N} \) with independent inter-arrival times \( X : \Omega \to \mathbb{R}_+^\mathbb{N} \) with first inter-renewal time distribution \( G \) and common inter-renewal time distribution \( F \) for \( \{X_n : n \geq 2\} \). Let the renewal function be denoted by \( m_0(t) \) and means \( \mathbb{E}X_1 = \mu_G \) and \( \mathbb{E}X_2 = \mu_F \). For any directly Riemann integrable function \( z \in \mathcal{D} \) and \( F \) non-lattice, we have

\[
\lim_{t \to \infty} \int_0^t z(t-x)dm_0(x) = \frac{1}{\mu_F} \int_0^\infty z(t)dt.
\]

**Remark 2.** Any kernel function \( K(t) = P \{Z_t \in A, X_1 > t\} \leq \bar{F}(t) \), and hence is d.R.i. from Proposition A.3(b).

**Example 1.4 (Limiting distribution of regenerative process).** For a regenerative process \( Z \) over a delayed renewal process \( S \) with finite mean i.i.d. inter-arrival times, we have \( K_2(t) = P \{Z_{t+1} \in A, X_2 > t\} \leq \bar{F}(t) \) for any \( A \in \mathcal{B}(\mathbb{R}) \), and hence the kernel function \( K_2 \in \mathcal{D} \). Applying Key Renewal Theorem to renewal function, we get the limiting probability of the event \( \{Z_t \in A\} \) as

\[
\lim_{t \to \infty} P \{Z_t \in A\} = \lim_{t \to \infty} (m_D * K_2)(t) = \frac{1}{\mu_F} \int_0^\infty K_2(t)dt.
\]
Example 1.5 (Limiting distribution of age and excess time). For a delayed renewal process $S$ with finite mean independent inter-renewal times such that the distribution of first renewal time is $F$, and the distribution of subsequent renewal times are identically $F$. Denoting the associated counting process by $N_D$ and renewal function $m_D$, we can write the limiting probability distribution of age as $F_e(x) = \lim_{t \to \infty} P\{A(t) \leq x\}$. We can write the complementary distribution as

$$F_e(x) = \lim_{t \to \infty} P\{A(t) > t\} = \lim_{t \to \infty} \int_0^t dm_D(t-y) I_{[y,x)} F(y) = \frac{1}{\mu_F} \int_x^\infty F(y) dy.$$

Example 1.6 (Limiting on probability of alternating renewal process). Consider an alternating renewal process $W$ with random on and off time sequence $Z$ and $Y$ respectively, such that $(Z,Y)$ is i.i.d. We denote the distribution of on and off times by non-lattice functions $H$ and $G$ respectively. If $EZ_n$ and $EY_n$ are finite, then applying Key renewal theorem to the limiting probability of alternating process being on, we get

$$\lim_{t \to \infty} P(t) = \lim_{t \to \infty} (m \ast \bar{H})(t) = \frac{EZ_n}{EZ_n + EY_n}.$$

A Directly Riemann Integrable

For each scalar $h > 0$ and natural number $n \in \mathbb{N}$, we can define intervals $I_n(h) \triangleq [(n-1)h, nh)$, such that the collection $(I_n(h), n \in \mathbb{N})$ partitions the positive real-line $\mathbb{R}_+$. For any function $z : \mathbb{R}_+ \to \mathbb{R}_+$ be a function bounded over finite intervals, we can denote the infimum and supremum of $z$ in the interval $I_n(h)$ as

$$z_i(n) \triangleq \inf\{z(t) : t \in I_n(h)\} \quad \text{and} \quad z_s(n) \triangleq \sup\{z(t) : t \in I_n(h)\}.$$

We can define functions $z_i, z_s : \mathbb{R}_+ \to \mathbb{R}_+$ such that $z_i(t) \triangleq \sum_{n \in \mathbb{N}} z_i(n) I_{I_n(h)}(t)$ and $z_s(t) \triangleq \sum_{n \in \mathbb{N}} z_s(n) I_{I_n(h)}(t)$ for all $t \in \mathbb{R}_+$. From the definition, we have $z_i \leq z \leq z_s$ for all $h > 0$. The infinite sums of infimum and supremums over all the intervals $(I_n(h), n \in \mathbb{N})$ are denoted by

$$\int_{\mathbb{R}_+} z_i(n) dt = h \sum_{n \in \mathbb{N}} z_i(n), \quad \int_{\mathbb{R}_+} z_s(n) dt = h \sum_{n \in \mathbb{N}} z_s(n).$$

Remark 3. Since $z_i \leq z \leq z_s$, we observe that $\int_{\mathbb{R}_+} z_i(n) dt \leq \int_{\mathbb{R}_+} z(t) dt \leq \int_{\mathbb{R}_+} z_s(n) dt$. If both left and right limits exist and are equal, then the integral value $\int_{\mathbb{R}_+} z(t) dt$ is equal to the limit.

Definition A.1 (directly Riemann integrable (d.R.i.)). A function $z : \mathbb{R}_+ \to \mathbb{R}_+$ is directly Riemann integrable and denoted by $z \in \mathcal{D}$ if the partial sums obtained by summing the infimum and supremum of $h$, taken over intervals obtained by partitioning the positive axis, are finite and both converge to the same limit, for all finite positive interval lengths. That is,

$$\lim_{h \to 0} \int_{\mathbb{R}_+} z_i(n) dt = \lim_{h \to 0} \int_{\mathbb{R}_+} z_s(n) dt.$$

The limit is denoted by $\int_{\mathbb{R}_+} z(t) dt$.

For a real function $z : \mathbb{R}_+ \to \mathbb{R}$, we can define the positive and negative parts by $z^+, z^- : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $t \in \mathbb{R}_+$, $z^+(t) \triangleq z(t) \vee 0$, and $z^-(t) \triangleq -z(t) \wedge 0$. If both $z^+, z^- \in \mathcal{D}$, then $z \in \mathcal{D}$ and the limit is

$$\int_{\mathbb{R}_+} z(t) dt \triangleq \int_{\mathbb{R}_+} z^+(t) dt - \int_{\mathbb{R}_+} z^-(t) dt.$$

Remark 4. We compare the definitions of directly Riemann integrable and Riemann integrable functions. For a finite positive $M$, a function $z : [0,M] \to \mathbb{R}$ is Riemann integrable if

$$\lim_{h \to 0} \int_0^M z_i(t) dt = \lim_{h \to 0} \int_0^M z_s(t) dt.$$
In this case, the limit is the value of the integral \( \int_{0}^{M} z(t) dt \). For a function \( z : \mathbb{R}_+ \to \mathbb{R} \),

\[
\int_{t \in \mathbb{R}_+} z(t) dt = \lim_{M \to \infty} \int_{0}^{M} z(t) dt,
\]

if the limit exists. For many functions, this limit may not exist.

**Remark 5.** A directly Riemann integrable function over \( \mathbb{R}_+ \) is also Riemann integrable, but the converse need not be true. For instance, for \( E_n \triangleq \left[ n - \frac{1}{2n^2}, n + \frac{1}{2n^2} \right] \) for each \( n \in \mathbb{N} \), consider the following Riemann integrable function \( z : \mathbb{R}_+ \to \mathbb{R}_+ \)

\[
z(t) = \sum_{n \in \mathbb{N}} 1_{E_n}(t), \quad t \in \mathbb{R}_+.
\]

We observe that \( z \) is Riemann integrable, but \( \int_{t \in \mathbb{R}_+} z_h(t) dt \) is always infinite for every \( h > 0 \).

**Proposition A.2 (Necessary conditions for d.R.i.).** If a function \( z : \mathbb{R}_+ \to \mathbb{R}_+ \) is directly Riemann integrable, then \( z \) is bounded and continuous a.e.

**Proposition A.3 (Sufficient conditions for d.R.i.).** A function \( z : \mathbb{R}_+ \to \mathbb{R}_+ \) is directly Riemann integrable, if any of the following conditions hold.

(a) \( z \) is monotone non-increasing, and Lebesgue integrable.

(b) \( z \) is bounded above by a directly Riemann integrable function.

(c) \( z \) has bounded support.

(d) \( \int_{t \in \mathbb{R}_+} z_h dt \) is bounded for some \( h > 0 \).

**Proposition A.4 (Tail Property).** If \( z : \mathbb{R}_+ \to \mathbb{R}_+ \) is directly Riemann integrable and has bounded integral value, then \( \lim_{t \to \infty} z(t) = 0 \).

**Corollary A.5.** Any distribution \( F : \mathbb{R}_+ \to [0,1] \) with finite mean \( \mu \), the complementary distribution function \( \bar{F} \) is d.R.i.

**Proof.** Since \( \bar{F} \) is monotonically non-increasing and its Lebesgue integration is \( \int_{\mathbb{R}_+} \bar{F}(t) dt = \mu \), the result follows from Proposition A.3(a).

**B Chebyshev’s sum inequality**

**Lemma B.1.** Let \( f : \mathbb{R} \to \mathbb{R}_+ \) and \( g : \mathbb{R} \to \mathbb{R}_+ \) be arbitrary functions with the same monotonicity. For any random variable \( X \), functions \( f(X) \) and \( g(X) \) are positive and

\[
\mathbb{E}[f(X)g(X)] \geq \mathbb{E}[f(X)]\mathbb{E}[g(X)]
\]

**Proof.** Let \( Y \) be a random variable independent of \( X \) and with the same distribution. Then,

\[
(f(X) - f(Y))(g(X) - g(Y)) \geq 0.
\]

Taking expectation on both sides the result follows.