Lecture-12: Applications of Key Renewal Theorem

1 Age-dependent branching process

Suppose a population where each organism lives for an *i.i.d.*random time period of T units with common distribution function F. Just before dying, each organism produces a number of offsprings N, an *i.i.d.*discrete random variable with common distribution P. Let $X(t)$ denote the number of organisms alive at time t. The stochastic process $X : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ is called an age-dependent branching process. We are interested in computing $M(t) = \mathbb{E}X(t)$ when $m = \mathbb{E}[N] = \sum_{j\in\mathbb{N}} jP_j$. This is a popular model in biology for population growth of various organisms.

Theorem 1.1. *If $X(0) = 1$, $m > 1$ and F is non lattice, then*

$$\lim_{t\to\infty} e^{-\alpha t} m(t) = \frac{n-1}{n^2\alpha \int_0^\infty xe^{-\alpha x} dF(x)},$$

where $\alpha > 0$ is the unique solution to the equation $n\int_0^\infty e^{-\alpha x} dF(x) = 1$.

Proof. Recall that for the delayed renewal function $m_D(t) \triangleq \sum_{n\in\mathbb{N}} (H * G^{(n-1)})(t)$ associated with delayed renewal time distribution $H(t) = 1 - e^{-\alpha t}\bar{F}(t)$ and inter-renewal distribution $G(t) = n\int_0^t e^{-\alpha u} dF(u)$, where α is the unique solution of

$$1 = n\int_0^\infty e^{-\alpha t} dF(t),$$

we can write the mean $m(t) = \mathbb{E}X(t)$ as the solution to delayed renewal equation as

$$m(t)e^{-\alpha t} = e^{-\alpha t}\bar{F}(t) + \int_0^t e^{-\alpha(t-u)}\bar{F}(t-u) dm_D(u).$$

Since $\bar{H}(t) = e^{-\alpha t}\bar{F}(t)$ is non-negative, monotone non-increasing and integrable, it is directly Riemann integrable. Hence, we can apply key renewal theorem to the limiting value of solution to renewal equation to obtain

$$\lim_{t\to\infty} m(t)e^{-\alpha t} = \frac{1}{\mu_G}\int_0^\infty e^{-\alpha t}\bar{F}(t)dt = \frac{\int_0^\infty e^{-\alpha t}\bar{F}(t)dt}{n\int_0^\infty te^{-\alpha t} dF(t)}.$$

Result follows from the integration by parts,

$$\int_0^\infty e^{-\alpha t}\bar{F}(t)dt = \frac{1}{\alpha} - \frac{1}{\alpha}\int_0^\infty e^{-\alpha t} dF(t) = \frac{1}{\alpha}\left(1 - \frac{1}{n}\right).$$

□

2 Equilibrium renewal process

Recall that the limiting distribution of age for a renewal process is given by the **equilibrium distribution** $F_e : \mathbb{R}_+ \to [0,1]$ defined for an inter-renewal time distribution F as $F_e(x) = \frac{1}{\mu_F}\int_0^x \bar{F}(y)dy$ for all $x \geqslant 0$.

Lemma 2.1. *The moment generating function of $F_e(x)$ is $\tilde{F}_e(s) = \frac{1-\tilde{F}(s)}{s\mu}$.*

Proof. By definition, $\tilde{F}_e(s) = \mathbb{E}\left[e^{-sX}\right]$, where X is a random variable with distribution function $F_e(x)$. We use integration by parts, to write

$$\tilde{F}_e(s) = \int_0^\infty e^{-sx} dF_e(x) = \frac{1}{s\mu} - \frac{1}{s\mu}\int_0^\infty e^{-sx} dF(x) = \frac{1}{s\mu}(1 - \tilde{F}(s)).$$

□

A delayed renewal process with the initial arrival distribution $G = F_e$ is called the **equilibrium renewal process**. Observe that F_e is the limiting distribution of the age and the excess time for the renewal process with common inter-renewal distribution F. Hence, if we start observing a renewal process at some arbitrarily large time t, then the observed renewal process is the equilibrium renewal process. This delayed renewal process exhibits stationary properties. That is, the limiting behaviors are exhibited for all times.

Theorem 2.2 (renewal function). *The renewal function $m_e(t)$ for the equilibrium renewal process is linear for all times. That is, $m_e(t) = \frac{t}{\mu}$.*

Proof. We know that the Laplace transform of renewal function $m_e(t)$ is given by

$$\tilde{m}_e(s) = \frac{\tilde{G}(s)}{1-\tilde{F}(s)} = \frac{\tilde{F}_e(s)}{1-\tilde{F}(s)} = \frac{1}{s\mu}. \tag{1}$$

Further, we know that the Laplace transform of function t/μ is given by $\mathcal{L}_{t/\mu}(s) = \frac{1}{\mu}\int_0^\infty e^{-sx}dx = \frac{1}{s\mu}$. Since moment generating function is a one-to-one map, $m_e(t) = \frac{t}{\mu}$ is the unique renewal function. □

Theorem 2.3 (excess time). *The distribution of excess time $Y_e(t)$ for the equilibrium renewal process is stationary. That is,*

$$P\{Y_e(t) \leqslant x\} = F_e(x),\ t \geqslant 0. \tag{2}$$

Proof. Since the excess time $Y_e(t)$ is regenerative process and $dm_e(t) = 1/\mu$, we can write

$$P\{Y_e(t) > x\} = \bar{F}_e(t+x) + \frac{1}{\mu}\int_0^t \bar{F}(t+x-u)du = \bar{F}_e(t+x) + \frac{1}{\mu}\int_x^{t+x} \bar{F}(y)dy = \bar{F}_e(x). \tag{3}$$

□

When we start observing the counting process at time s, the observed renewal process is delayed renewal process with initial distribution Y_e at time s being identical to the distribution F_e. Hence, the number of renewals $N_e(t+s) - N_e(s)$ has the same distribution as $N_e(t)$ in duration t. That is, the distribution of counting process is shift invariant.

Theorem 2.4 (stationary increments). *The counting process $N_e : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ for the equilibrium renewal process has stationary increments.*

Proof. The Laplace transform of $N_e(t+s) - N_e(s)$ is identical to Laplace transform of $N_e(t)$. Result holds from the uniqueness of inverse of Laplace transforms. □

Example 2.5 (Poisson process). Consider the case, when inter-renewal time distribution F for a delay renewal process is exponential with rate λ. Here, one would expect the equilibrium distribution $F_e = F$, since Poisson process has stationary and independent increments. We observe that

$$F_e(x) = \frac{1}{\mu}\int_0^x \bar{F}(y)dy = \lambda\int_0^x \mathrm{e}^{-\lambda y}dy = 1 - e^{-\lambda x} = F(x).$$

We see that F_e is also distributed exponentially with rate λ. Indeed, this is a Poisson process with rate λ.