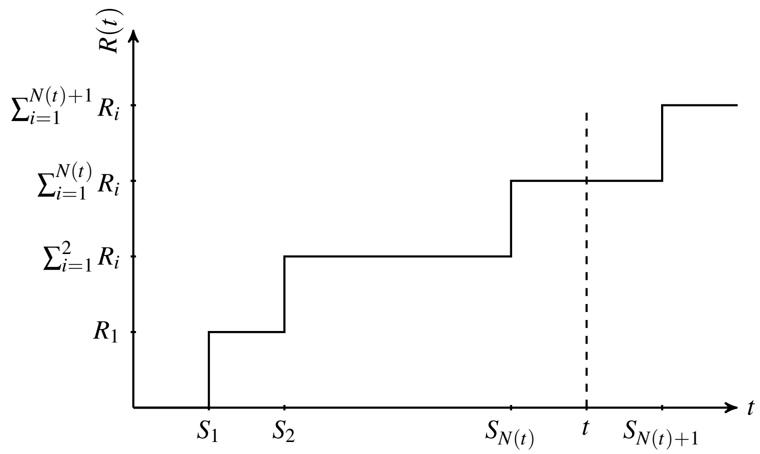
Lecture-13: Renewal Reward Processes

1 Renewal reward process

Consider a counting process $(N(t) : t \geqslant 0)$ associated with *iid* inter renewal times $(X_n : n \in \mathbb{N})$ having common distribution F. At the end of nth renewal interval, a random reward R_n is earned for each $n \in \mathbb{N}$. Let (X_n, R_n) be *iid* with the reward R_n earned in nth renewal possibly dependent on the duration X_n. Then the **reward process** $(R(t) : t \geqslant 0)$ consists of accumulated reward earned by time t as $R(t) = \sum_{i=1}^{N(t)} R_i$.



Theorem 1.1 (renewal reward). *Let the mean of absolute value of reward $\mathbb{E}|R_n|$, and mean of absolute value of renewal duration $\mathbb{E}|X_n|$ be finite. Then the empirical average of reward converges, almost surely and in mean, i.e.*

$$\lim_{t\to\infty} \frac{R(t)}{t} = \frac{\mathbb{E}R_n}{\mathbb{E}X_n} \text{ a.s. }, \qquad\qquad \lim_{t\to\infty} \frac{\mathbb{E}R(t)}{t} = \frac{\mathbb{E}R}{\mathbb{E}X}.$$

Proof. We can write the rate of accumulated reward as $\frac{R(t)}{t} = (\frac{R(t)}{N(t)})(\frac{N(t)}{t})$. From the strong law of large numbers and the fact that $\lim_{t\to\infty} N(t) = \infty$ almost surely, we obtain $\lim_{t\to\infty} \frac{\sum_{i=1}^{N(t)} R_i}{N(t)} = \mathbb{E}R_1$. From the strong law for counting processes we have $\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mathbb{E}X_1}$ almost surely.

Since $N(t)+1$ is a stopping time for the sequence $\{(X_1,R_1),(X_2,R_2),\dots\}$, by Wald's lemma,

$$\mathbb{E}R(t) = \mathbb{E}\left[\sum_{i=1}^{N(t)} R_i\right] = \mathbb{E}\left[\sum_{i=1}^{N(t)+1} R_i\right] - \mathbb{E}R_{N(t)+1} = (m(t)+1)\mathbb{E}R_1 - \mathbb{E}R_{N(t)+1}.$$

Defining $g(t) \triangleq \mathbb{E}R_{N(t)+1}$, using elementary renewal theorem, it suffices to show that $\lim_{t\to\infty} g(t)/t = 0$. Observe that $R_{N(t)+1}$ is a regenerative process with the regenerative sequence being the renewal instants. We can write the kernel function as

$$K(t) \triangleq \mathbb{E}[R_{N(t)+1}\mathbb{1}_{\{X_1>t\}}] = \mathbb{E}[R_1\mathbb{1}_{\{X_1>t\}}] \leqslant \mathbb{E}|R_1|.$$

Using the solution to renewal function, we can write $g = (1+m)*K$ in terms of renewal function m and kernel function K. From finiteness of $\mathbb{E}|R|$, it follows that $\lim_{t\to\infty} K(t) = 0$, and we can choose T such that $|K(u)| \leq \varepsilon$ for all $u \geq T$. Hence, for all $t \geq T$, we have

$$\begin{aligned}\frac{|g(t)|}{t} &\leq \frac{|K(t)|}{t} + \int_0^{t-T} \frac{|K(t-u)|}{t} dm(u) + \int_{t-T}^{t} \frac{|K(t-u)|}{t} dm(u)\\ &\leq \frac{\varepsilon}{t} + \frac{\varepsilon m(t-T)}{t} + \mathbb{E}|R_1| \frac{(m(t)-m(t-T))}{t}.\end{aligned}$$

Taking limits and applying elementary renewal and Blackwell's theorem, we get

$$\limsup_{t\to\infty}\frac{|g(t)|}{t}\leq\frac{\varepsilon}{\mathbb{E}X}.$$

The result follows since $\varepsilon > 0$ was arbitrary. □

Lemma 1.2. *Reward $R_{N(t)+1}$ at the next renewal has different distribution than R_1.*

Proof. Notice that $R_{N(t)+1}$ is related to $X_{N(t)+1}$ which is the length of the renewal interval containing the point t. We have seen that larger renewal intervals have a greater chance of containing t. That is, $X_{N(t)+1}$ tends to be larger than a ordinary renewal interval. Since $R_{N(t)+1}$ is a regenerative process, we can formally write its tail probability as

$$f(t) = P\{R_{N(t)+1} > x\} = K(t) + (m * K)(t),$$

where in terms of the distribution functions F, H for inter-renewal times and rewards we can write the the kernel

$$K(t) = P\{R_{N(t)+1} > x, X_1 > t\} = P\{R_1 > x, X_1 > t\} \leqslant \bar{F}(t).$$

It follows that $f(t) \leqslant \bar{F}(t) + (m * \bar{F})(t) = 1$. □

Lemma 1.3. *Renewal reward theorem applies to a reward process $R(t)$ that accrues reward continuously over a renewal duration. The total reward in a renewal duration X_n remains R_n as before, with the sequence$((X_n, R_n) : n \in \mathbb{N})$ being iid.*

Proof. Let the process $R(t)$ denote the accumulated reward till time t, when the reward accrual is continuous in time. Then, it follows that

$$\frac{\sum_{n=1}^{N(t)} R_n}{t} \leq \frac{R(t)}{t} \leq \frac{\sum_{n=1}^{N(t)+1} R_n}{t}.$$

Result follows from application of strong law of large numbers. □

1.1 Limiting empirical average of age and excess times

To determine the average value of the age of a renewal process, consider the following gradual reward process. We assume the reward rate to be equal to the age of the process at any time t, and

$$R(t) = \int_0^t A(u)du.$$

Observe that age is a linear increasing function of time in any renewal duration. In nth renewal duration, it increases from 0 to X_n, and the total reward $R_n = X_n^2/2$. Hence, we obtain from the renewal reward theorem

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t A(u)du = \frac{\mathbb{E}R_n}{\mathbb{E}X_n} = \frac{\mathbb{E}X^2}{2\mathbb{E}X}.$$

Example 1.4. Since the accumulated excess time during one renewal cycle is $\int_0^{X_n}(X_n - t)dt$, the limiting empirical average of excess time $Y(t) = t - S_{N(t)}$ can be found using the renewal reward theorem is

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t Y(u)du = \frac{\mathbb{E}[X^2]}{2\mathbb{E}[X]}.$$

Example 1.5. The limiting average of current renewal interval $X_{N(t)} = A(t) + Y(t) = S_{N(t)+1} - S_{N(t)}$ can be computed directly as the sum of two limiting averages, or from the application of renewal reward theorem with accrued reward in one renewal interval being $\int_0^{X_n} X_n dt = X_n^2$, to get

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t X_{N(u)+1}du = \frac{\mathbb{E}[X^2]}{\mathbb{E}[X]}.$$

We see that this limit is always greater than $\mathbb{E}[X]$, except when X is constant. Such a result was to be expected in view of the inspection paradox, since we can show that $\lim_{t\to\infty}\mathbb{E}[X_{N(t)+1}] = \lim_{t\to\infty}\frac{1}{t}\int_0^t X_{N(u)+1}du.$

Example 1.6. It can be shown, under certain regularity conditions, that

$$\lim_{t\to\infty}\mathbb{E}R_{N(t)+1} = \lim_{t\to\infty}\frac{1}{t}\int_0^t R_{N(u)+1}du = \frac{\mathbb{E}[R_1X_1]}{\mathbb{E}[X_1]}.$$

If reward is a monotonically increasing function of renewal interval, then we get that $\lim_{t\to\infty}\mathbb{E}R_{N(t)+1} \geqslant \mathbb{E}R_1$ from Chebyshev's inequality.

1.2 Stationary probability and empirical average

Theorem 1.7. *For an alternative renewal process $W : \Omega \to \{0,1\}^{\mathbb{R}_+}$ the stationary probability of being on is same as the limiting average time spent in the on duration if the renewal duration has finite mean. That is,*

$$\lim_{t\to\infty} P\{W(t) = 1\} = \lim_{t\to\infty} \frac{1}{t}\int_0^t W(u)du.$$

Proof. Suppose for an alternating renewal process, we earn at a unit rate in on state. The aggregate reward in one renewal duration X_n is the on time Z_n in that duration.

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t W(u)du = \lim_{t\to\infty} \frac{R(t)}{t} = \frac{\mathbb{E}Z_n}{\mathbb{E}X_n} = \lim_{t\to\infty} P(\text{on at time } t).$$

□

A Chebyshev's sum inequality

Lemma A.1. *Let $f : \mathbb{R} \to \mathbb{R}_+$ and $g : \mathbb{R} \to \mathbb{R}_+$ be arbitrary functions with the same monotonicity. For any random variable X, functions $f(X)$ and $g(X)$ are positive and*

$$\mathbb{E}[f(X)g(X)] \geq \mathbb{E}[f(X)]\mathbb{E}[g(X)].$$

Proof. Let Y be a random variable independent of X and with the same distribution. Then,

$$(f(X) - f(Y))(g(X) - g(Y)) \geq 0.$$

Taking expectation on both sides the result follows. □