Lecture-21: Reversible Processes and Queues

1 Equilibrium distribution of M/M/1 Queue

Recall the global balance equations for equilibrium distribution $\pi \in [0,1]^{\mathbb{Z}_+}$ are

$$-\pi_0\lambda + \pi_1\mu = 0, \qquad\qquad \pi_{k-1}\lambda - \pi_k(\lambda+\mu) + \pi_{k+1}\mu = 0, \quad k \in \mathbb{N}.$$

Recognizing that π is a one-sided countably infinite sequence, we denote the discrete Fourier transform or the z-transform of the distribution $\pi \in [0,1]^{\mathbb{Z}_+}$ as

$$\Pi(z) \triangleq \sum_{k\in\mathbb{Z}_+} \pi_k z^k.$$

Using this notation, we can compute

$$(\lambda+\mu)\sum_{k\in\mathbb{N}} \pi_k z^k = \lambda \sum_{k\in\mathbb{N}} \pi_{k-1} z^k + \mu \sum_{k\in\mathbb{N}} \pi_{k+1} z^k.$$

Using the definition of $\Pi(z)$, we can re-write this as

$$(\lambda+\mu)(\Pi(z)-\pi_0) = \lambda z \Pi(z) + \mu z^{-1}(\Pi(z) - \pi_0 - z\pi_1).$$

Since $\pi_1 = \pi_0\rho$ where $\rho = \frac{\lambda}{\mu}$, we can re-arrange the terms to get

$$\Pi(z) = \frac{\pi_0}{(1-\rho z)}.$$

Inverting the z-transform, we get

$$\pi_k = \pi_0\rho^k, \quad k \in \mathbb{N}.$$

2 Reversed Processes

Definition 2.1. Let $X : \Omega \to \mathcal{X}^T$ be a stochastic process with index set T being an additive ordered group such as $\mathbb{R}$ or $\mathbb{Z}$. Then, $\hat{X}^\tau : \Omega \to \mathcal{X}^T$ defined as $\hat{X}^\tau(t) \triangleq X(\tau - t)$ for all $t \in T$ is the **reversed process** for some $\tau \in T$.

Remark 1. Note that a reversed process, doesn't have to have the identical distribution to the original process. For a reversible process X, the reversed process would have identical distribution.

Lemma 2.2. *If $X : \Omega \to \mathcal{X}^T$ is a Markov process, then the reversed process $\hat{X}^\tau$ is also Markov for any $\tau \in T$.*

Proof. Let $\mathcal{F}_t = \sigma(X(s) : s \leqslant t)$ denote the history of the process until time t. From the Markov property of process X, we have for any event $B \in \mathcal{F}_{t+u}$, states $x, y \in \mathcal{X}$ and times $u, s > 0$

$$P(B | \{X_t = y, X_{t-s} = x\}) = P(B | \{X_t = y\}).$$

Markov property of the reversed process follows from the observation, that

$$P(\{X_{t-s} = x\} \,|\, \{X_t = y\} \cap B) = \frac{P(\{X_{t-s} = x, X_t = y\} \cap B)}{P(\{X_t = y\} \cap B)} = \frac{P\{X_{t-s} = x, X_t = y\} P(B | \{X_{t-s} = x, X_t = y\})}{P\{X_t = y\} P(B | \{X_t = y\})} = P(\{X_{t-s} = x\} \,|\, \{X_t = y\}).$$

□

Remark 2. Even if the forward process X is time-homogeneous, the reversed process need not be time-homogeneous. For a non-stationary time-homogeneous Markov process, the reversed process is Markov but not necessarily time-homogeneous.

Theorem 2.3. *If $X : \Omega \to \mathcal{X}^{\mathbb{R}}$ is an irreducible, positive recurrent, stationary, and homogeneous Markov process with transition kernel P and equilibrium distribution π, then the reversed Markov process $\hat{X}^{\tau} : \Omega \to \mathcal{X}^{\mathbb{R}}$ is also irreducible, positive recurrent, stationary, and homogeneous with the same equilibrium distribution π and transition kernel $\hat{P}$ given*

$$\hat{P}_{xy}(t) \triangleq \frac{\pi_y}{\pi_x} P_{yx}(t) \text{ for all } t \in T, \text{ and states } x, y \in \mathcal{X}.$$

Further, for any finite sequence $x \in \mathcal{X}^n$, we have

$$P_{\pi} \cap_{i=1}^{n} \{X_{t_i} = x_i\} = \hat{P}_{\pi} \cap_{i=1}^{n} \left\{\hat{X}_{t_i} = x_{n-i+1}\right\}.$$

Proof. We can check that $\hat{P}$ is a probability transition kernel, since $\hat{P}_{xy} \geqslant 0$ for all $t \in T$ and

$$\sum_{y \in \mathcal{X}} \hat{P}_{xy}(t) = \frac{1}{\pi_x} \sum_{y \in \mathcal{X}} \pi_y P_{yx}(t) = 1.$$

Further, we see that π is an invariant distribution for $\hat{P}$, since for all states $x, y \in \mathcal{X}$

$$\sum_{x \in \mathcal{X}} \pi_x \hat{P}_{xy}(t) = \pi_y \sum_{x \in \mathcal{X}} P_{yx}(t) = \pi_y.$$

We next wish to show that $\hat{P}$ defined in the Theorem, is the probability transition kernel for the reversed process. Since the forward process is stationary and time-homogeneous, we can write the probability transition kernel for the reversed process as

$$P(\{\hat{X}^{\tau}_{\tau-t+s} = x\} \mid \{\hat{X}^{\tau}_{\tau-t} = y\}) = \frac{P\left\{\hat{X}^{\tau}_{\tau-t+s} = x, \hat{X}^{\tau}_{\tau-t} = y\right\}}{P\left\{\hat{X}^{\tau}_{\tau-t} = y\right\}} = \frac{P_{\pi}\left\{X_{t-s} = x, X_t = y\right\}}{P_{\pi}\left\{X_t = y\right\}} = \frac{\pi_x P_{xy}(0,s)}{\pi_y}.$$

This implies that the reversed process is time-homogeneous and has the desired probability transition kernel. Further, π is the stationary distribution for the reversed process and is the marginal distribution for the reversed process at any time t, and hence the reversed process is also stationary.

For an irreducible and positive recurrent Markov process with stationary distribution π, we have $\pi_x > 0$ for each state $x \in \mathcal{X}$. Since the forward process is irreducible, there exists a time $t \geqslant 0$ such that $P_{yx}(t) > 0$ for states $x, y \in \mathcal{X}$, and hence $\hat{P}_{xy}(t) > 0$ implying irreducibility of the reversed process. From the Markov property of the underlying processes and definition of $\hat{P}$, we can write

$$P_{\pi}\{X_{t_1} = x_1, \dots, X_{t_n} = x_n\} = \pi_{x_1} \prod_{i=1}^{n-1} P_{x_i x_{i+1}}(t_{i+1} - t_i) = \pi_{x_n} \prod_{i=1}^{n-1} \hat{P}_{x_{i+1} x_i}(t_{i+1} - t_i) = \hat{P}_{\pi}\{\hat{X}_{t_1} = x_n, \dots, \hat{X}_{t_n} = x_1\}.$$

This follows from the fact that

$$\pi_{x_1} P_{x_1 x_2}(t_2 - t_1) = \pi_{x_2} \hat{P}_{x_2 X_1}(t_2 - t_1),$$

and hence we have

$$\pi_{x_1} \prod_{i=1}^{n-1} P_{x_i x_{i+1}}(t_{i+1} - t_i) = \pi_{x_n} \prod_{i=1}^{n-1} \hat{P}_{x_{i+1} x_i}(t_{i+1} - t_i).$$

Let's take $\tau = t_n + t_1$, then we have $\hat{X}^{\tau}_t = X(t_n + t_1 - t)$ and hence we have $(X_{t_1}, \dots, X_{t_i}, \dots, X_{t_n}) = (\hat{X}^{\tau}_{t_n}, \dots, \hat{X}^{\tau}(t_1 + t_n - t_i), \dots, \hat{X}^{\tau}_{t_1})$. From the Markovity of the reversed process, we can write

$$\hat{P}_{\pi}\left\{\hat{X}^{\tau}_{t_n} = x_1, \dots, \hat{X}^{\tau}_{t_1} = x_n\right\} = \hat{P}_{\pi}\left\{\hat{X}^{\tau}_{t_1} = x_n, \dots, \hat{X}^{\tau}_{t_n} = x_1\right\} = \pi_{x_n} \prod_{i=1}^{n-1} \hat{P}(\hat{X}^{\tau}_{\tau - t_{n-i}} = x_{n-i} | \hat{X}^{\tau}_{\tau - t_{n-i+1}} = x_{n-i+1})$$

$$= \pi_{x_n} \prod_{i=1}^{n-1} \hat{P}_{x_{n-i+1} x_{n-i}}(t_{n-i+1} - t_{n-i}) = \pi_{x_n} \prod_{i=1}^{n-1} \hat{P}_{x_{i+1} x_i}(t_{i+1} - t_i).$$

For any finite $n \in \mathbb{N}$, we see that the joint distributions of $(X_{t_1}, \dots, X_{t_n})$ and $(X_{s+t_1}, \dots, X_{s+t_n})$ are identical for all $s \in T$, from the stationarity of the process X. It follows that $\hat{X}$ is also stationary, since $(\hat{X}_{t_n}, \dots, \hat{X}_{t_1})$ and $(\hat{X}_{s+t_n}, \dots, \hat{X}_{s+t_1})$ have the identical distribution. ☐

Corollary 2.4. *If $X : \Omega \to \mathcal{X}^{\mathbb{Z}}$ is an irreducible, stationary, homogeneous Markov chain with transition matrix P and equilibrium distribution π, then the reversed chain $\hat{X}^{\tau} : \Omega \to \mathcal{X}^{\mathbb{Z}}$ is an irreducible stationary, time homogeneous Markov chain with the same equilibrium distribution π, and transition matrix $\hat{P}$ given by*

$$\hat{P}_{xy} = \frac{\pi_y}{\pi_x} P_{yx}.$$

Corollary 2.5. *If $X : \Omega \to \mathcal{X}^{\mathbb{R}}$ is an irreducible, stationary, homogeneous Markov process with generator matrix Q and equilibrium distribution π, then the reversed process $\hat{X}^{\tau} : \Omega \to \mathcal{X}^{\mathbb{R}}$ is also an irreducible, stationary, homogeneous Markov process with same equilibrium distribution π and generator matrix $\hat{Q}$ such that*

$$\hat{Q}_{xy} = \frac{\pi_y}{\pi_x} Q_{yx}.$$

Proof. □

Corollary 2.6. *Consider irreducible Markov chain with transition matrix $P : \mathcal{X} \times \mathcal{X} \to [0,1]$. If one can find a non-negative vector $\alpha \in [0,1]^{\mathcal{X}}$ and other transition matrix $P^* : \mathcal{X} \times \mathcal{X} \to [0,1]$ such that $\sum_{x\in\mathcal{X}} \alpha_x = 1$ and satisfies the detailed balance equation*

$$\alpha_x P_{xy} = \alpha_y P^*_{yx},$$

then α is the stationary probability vector of P and P^* is the transition matrix for the reversed chain.

Proof. Summing $\alpha_i P_{ij} = \alpha_j P^*_{ji}$ over i gives, $\sum_i \alpha_i P_{ij} = \alpha_j$. Hence α_is are the stationary probabilities of the forward and reverse process. Since $P^*_{ji} = \frac{\alpha_i P_{ij}}{\alpha_j}$, P^*_{ij} are the transition probabilities of the reverse chain. □

Corollary 2.7. *Let $Q : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ denote the rate matrix for an irreducible Markov process. If we can find $Q^* : \mathcal{X} \times \mathcal{X} \to [0,1]$ and a vector $\pi \in [0,1]^{\mathcal{X}}$ such that $\sum_{x\in\mathcal{X}} \pi_x = 1$ and for $y \neq x \in \mathcal{X}$, we have*

$$\pi_x Q_{xy} = \pi_y Q^*_{yx}, \qquad \text{and} \qquad \sum_{y\neq x} Q_{xy} = \sum_{y\neq x} Q^*_{xy},$$

then Q^* is the rate matrix for the reversed Markov chain and π is the equilibrium distribution for both processes.

3 Applications of Reversed Processes

3.1 Truncated Markov Processes

Definition 3.1. For a Markov process $X : \Omega \to \mathcal{X}^{\mathbb{R}}$, and a subset $A \subseteq \mathcal{X}$ the boundary of A is defined as

$$\partial A \triangleq \left\{ y \notin A : Q_{xy} > 0, \text{ for some } x \in A \right\}.$$

Definition 3.2. Consider a transition rate matrix $Q : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ on the countable state space $\mathcal{X}$. Given a nonempty subset $A \subseteq \mathcal{X}$, the truncation of Q to A is the transition rate matrix $Q^A : A \times A \to \mathbb{R}$, where for all $x, y \in A$

$$Q^A_{xy} = \begin{cases} Q_{xy}, & y \neq x, \\ -\sum_{z\in A\setminus\{x\}} Q_{xz}, & y = x. \end{cases}$$

Proposition 3.3. *Suppose $X : \Omega \to \mathcal{X}^{\mathbb{R}}$ is an irreducible, time-reversible CTMC on the countable state space $\mathcal{X}$, with generator $Q : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ and stationary probabilities $\pi \in [0,1]^{\mathcal{X}}$. Suppose the trunctated Markov process to a set of states $A \subseteq \mathcal{X}$ is irreducible. Then, any stationary CTMC with state space A and generator Q^A is also time-reversible, with stationary probabilities*

$$\pi^A_y = \frac{\pi_y}{\sum_{x\in A} \pi_x}, \quad y \in A.$$

Proof. It is clear that π^A is a distribution on state space A. We must show the reversibility with this distribution π^A. That is, we must show for all $i, j \in A$

$$\pi^A_x Q_{xy} = \pi^A_y Q_{yx}.$$

However, this is true since the original chain is time reversible. □

Example 3.4 (Limiting waiting room: M/M/1/K). Consider a variant of the M/M/1 queueing system that has a finite buffer capacity of at most K customers. Thus, customers that arrive when there are already K customers present are 'rejected'. It follows that the CTMC for this system is simply the M/M/1 CTMC truncated to the state space $\{0, 1, \dots, K\}$, and so it must be time-reversible with stationary distribution

$$\pi_i = \frac{\rho^i}{\sum_{j=0}^{k}} \rho^j, \quad 0 \leqslant i \leqslant k.$$

Example 3.5 (Two queues with joint waiting room). Consider two independent M/M/1 queues with arrival and service rates λ_i and μ_i respectively for $i \in [2]$. Then, joint distribution of two queues is

$$\pi(n_1, n_2) = (1-\rho_1)\rho_1^{n_1}(1-\rho_2)\rho_2^{n_2}, \quad n_1, n_2 \in \mathbb{Z}_+.$$

Suppose both the queues are sharing a common waiting room, where if arriving customer finds R waiting customer then it leaves. In this case,

$$\pi(n_1, n_2) = \frac{\rho_1^{n_1} \rho_2^{n_2}}{\sum_{(m_1,m_2) \in A} \rho_1^{m_1} \rho_2^{m_2}}, \quad (n_1, n_2) \in A \subseteq \mathbb{Z}_+ \times \mathbb{Z}_+.$$