Lecture-24: Martingale Convergence Theorem

1 Martingale Convergence Theorem

Before we state and prove martingale convergence theorem, we state some results which will be used in the proof of the theorem.

Lemma 1.1. *If $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ is a submartingale and $T : \Omega \to \mathbb{N}$ is a stopping time with respect to a filtration $\mathcal{F}_\bullet$, such that $P\{T \leqslant n\} = 1$. Then*

$$\mathbb{E}X_1 \leqslant \mathbb{E}X_T \leqslant \mathbb{E}X_n.$$

Proof. Since T is bounded, it follows from Martingale stopping theorem, that $\mathbb{E}X_T \geqslant \mathbb{E}X_1$. Now, since T is a stopping time, we see that for the event $\{T = k\}$ for any $k \leqslant n$

$$\mathbb{E}[X_n \mathbb{1}_{\{T=k\}} | \mathcal{F}_k] \geqslant X_k \mathbb{1}_{\{T=k\}} = X_T \mathbb{1}_{\{T=k\}}.$$

Result follows by taking expectation on both sides and summing over k. That is,

$$\mathbb{E}X_n = \mathbb{E}\sum_{k=1}^{n} X_n \mathbb{1}_{\{T=k\}} \geqslant \mathbb{E}\sum_{k=1}^{n} X_T \mathbb{1}_{\{T=k\}} = \mathbb{E}X_T.$$

□

Definition 1.2. Consider a discrete random process $X : \Omega \to \mathbb{R}^{\mathbb{Z}_+}$ adapted to the filtration $\mathcal{F}_\bullet = (\mathcal{F}_n \subseteq \mathcal{F} : n \in \mathbb{Z}_+)$. For the two thresholds $a < b$, we define the stopping times corresponding to kth downcrossing and upcrossing times as

$$N_{2k-1} \triangleq \inf\{m > N_{2k-2} : X_m \leqslant a\}, \qquad N_{2k} \triangleq \inf\{m > N_{2k-1} : X_m \geqslant b\}.$$

We next define the indicator to the event that the process is in kth upcrossing transition from a to b at time m,

$$H_m \triangleq \sum_{k \in \mathbb{N}} \mathbb{1}_{\{N_{2k-1} < m \leqslant N_{2k}\}}.$$

The number of upcrossings completed in time n is defined by

$$U_n \triangleq \sup\{k \in \mathbb{N} : n \geqslant N_{2k}\}.$$

Remark 1. We can write $H_m = \sum_{k \in \mathbb{N}} \{m-1 \geqslant N_{2k-1}\} \cap \{m-1 \geqslant N_{2k}\}^c \in \mathcal{F}_{m-1}$, and hence the event that the process X is in an upcrossing transition at time m is predictable.

Lemma 1.3 (Upcrossing inequality). *Let $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ be a submartingale with respect to a filtration $\mathcal{F}_\bullet$, and define a random sequence $Y : \Omega \to \mathbb{R}^{\mathbb{N}}$ such that $Y_n \triangleq a + (X_n - a)^+ = X_n \vee a$ for each $n \in \mathbb{N}$. Then, we have*

$$(b-a)\mathbb{E}U_n \leqslant \mathbb{E}Y_n - \mathbb{E}Y_0.$$

Proof. Since X is a submartingale so is Y, as Y_n is a convex function of X_n. Since each upcrossing has a gain slightly more than $b-a$, the following inequality exists,

$$(b-a)U_n \leqslant \sum_{m=1}^{n} \sum_{k \in \mathbb{N}} \mathbb{1}_{\{N_{2k-1} < m \leqslant N_{2k}\}} (Y_{m+1} - Y_m) = (H \cdot Y)_n = \sum_{k=1}^{U_n} (Y_{N_{2k}} - Y_{N_{2k-1}}).$$

Let $K_m \triangleq 1 - H_m$ for each $m \in \mathbb{N}$. Since H is predictable, then so is K with respect to $\mathcal{F}_\bullet$, and

$$Y_n - Y_0 = (H \cdot Y)_n + (K \cdot Y)_n.$$

Since $H:\Omega\to\{0,1\}^{\mathbb{N}}$ is a non-negative and bounded sequence, so is $K:\Omega\to\{0,1\}^{\mathbb{N}}$. Further, since Y is a submartingale, so is $((K\cdot Y)_n : n\in\mathbb{Z}_+)$. Therefore, we can write

$$\mathbb{E}[(K\cdot Y)_n] \geqslant \mathbb{E}[(K\cdot Y)_0] = 0.$$

Therefore, it follows that

$$\mathbb{E}(Y_n - Y_0) = \mathbb{E}(H\cdot Y)_n + \mathbb{E}(K\cdot Y)_n \geqslant \mathbb{E}(H\cdot Y)_n \geqslant (b-a)\mathbb{E}U_n.$$

□

Theorem 1.4 (Martingale convergence theorem). *If $X:\Omega\to\mathbb{R}^{\mathbb{N}}$ is a submartingale with respect to filtration $\mathcal{F}_\bullet$ such that $\sup_{n\in\mathbb{N}}\mathbb{E}X_n^+ < \infty$, then $\lim_{n\in\mathbb{N}}X_n = X$ a.s with $\mathbb{E}|X| < \infty$.*

Proof. Since $(X-a)^+ \leqslant X^+ + |a|$, it follows from upcrossing inequality that

$$\mathbb{E}U_n \leqslant \frac{\mathbb{E}X_n^+ + |a|}{b-a}.$$

The number of upcrossings U_n increases with n, however the mean $\mathbb{E}U_n$ is bounded above for each $n\in\mathbb{N}$. Hence, $\lim_{n\in\mathbb{N}}\mathbb{E}U_n$ exists and is finite.

Let $U \triangleq \lim_{n\in\mathbb{N}} U_n$ and since $\mathbb{E}U \leqslant \sup_n \frac{\mathbb{E}X_n^+ + |a|}{b-a} < \infty$, we have $U<\infty$ almost surely. This conclusion implies

$$P_{a,b\in\mathbb{Q}}\cup\{\liminf_{n\in\mathbb{N}}X_n < a < b < \limsup_{n\in\mathbb{N}}X_n\} = 0.$$

From the above probability, we have almost sure equality

$$\limsup_{n\in\mathbb{N}}X_n = \liminf_{n\in\mathbb{N}}X_n.$$

That is, the limit $\lim_{n\in\mathbb{N}} X_n$ exists almost surely. Fatou's lemma guarantees

$$\mathbb{E}X^+ \leqslant \liminf_{n\in\mathbb{N}}\mathbb{E}X_n^+ < \infty,$$

which implies $X<\infty$ almost surely. From the submartingale property of X_n, it follows that

$$\mathbb{E}X_n^- = \mathbb{E}X_n^+ - \mathbb{E}X_n \leq \mathbb{E}X_n^+ - \mathbb{E}X_0.$$

From Fatou's lemma, we get

$$\mathbb{E}X^- \leqslant \liminf_{n\in\mathbb{N}}\mathbb{E}X_n^- \leqslant \sup_{n\in\mathbb{N}}\mathbb{E}X_n^+ - \mathbb{E}X_0 < \infty.$$

This implies $X > -\infty$ almost surely, completing the proof. □

Example 1.5 (Polya's Urn Scheme). Consider a discrete time stochastic process $((B_n,W_n) : n\in\mathbb{N})$, where B_n, W_n respectively denote the number of black and white balls in an urn after $n\in\mathbb{N}$ draws. At each draw n, balls are uniformly sampled from this urn. After each draw, one additional ball of the same color to the drawn ball, is returned to the urn. We are interested in characterizing evolution of this urn, given initial urn content (B_0,W_0). Let ξ_i be a random variable indicating the outcome of the ith draw being a black ball. For example, if the first drawn ball is a black, then $\xi_1 = 1$ and $(B_1,W_1) = (B_0+1,W_0)$. In general,

$$B_n = B_0 + \sum_{i=1}^n \xi_i = B_{n-1} + \xi_n, \qquad W_n = W_0 + \sum_{i=1}^n (1-\xi_i) = W_{n-1} + 1 - \xi_n.$$

It is clear that $B_n + W_n = B_0 + W_0 + n$. We are interested in limiting ratio of black balls. We represent the proportion of black balls after n draws by

$$X_n = \frac{B_n}{B_n + W_n} = \frac{B_n}{B_0 + W_0 + n}.$$

Let $\mathcal{F}_n = \sigma(B_0, W_0, \xi_1, \dots, \xi_n)$ be the σ-field generated by the first n indicators to black draws. It is clear that

$$\mathbb{E}[\xi_{n+1}|\mathcal{F}_n] = X_n.$$

Using this fact, we observe that $X : \Omega \to [0,1]^{\mathbb{N}}$ is a martingale adapted to filtration $\mathcal{F}_\bullet = (\mathcal{F}_n : n \in \mathbb{N})$, since

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = \frac{1}{B_0 + W_0 + n + 1}\mathbb{E}[B_{n+1}|\mathcal{F}_n] = \frac{B_n + X_n}{\frac{B_n}{X_n} + 1} = X_n.$$

For each $n \in \mathbb{N}$, we have $\mathbb{E}X_n^+ = \mathbb{E}X_n \leqslant 1$. From Martingale convergence theorem, it follows almost surely that

$$\lim_{n \in \mathbb{N}} X_n = X_0 = \frac{B_0}{B_0 + W_0}.$$