

Lecture-13: L^p convergence

1 L^p convergence

Definition 1.1 (Convergence in L^p). Let $p \geq 1$, then we say that a random sequence $X : \Omega \rightarrow \mathbb{R}^{\mathbb{N}}$ defined on a probability space (Ω, \mathcal{F}, P) **converges in L^p** to a random variable $X_\infty : \Omega \rightarrow \mathbb{R}$, if

$$\lim_n \|X_n - X_\infty\|_p = 0.$$

The convergence in L^p is denoted by $\lim_n X_n = X_\infty$ in L^p .

Remark 1. For $p \in [1, \infty)$, the convergence of a random sequence $X : \Omega \rightarrow \mathbb{R}^{\mathbb{N}}$ in L^p to a random variable $X_\infty : \Omega \rightarrow \mathbb{R}$ is equivalent to

$$\lim_n \mathbb{E} |X_n - X_\infty|^p = 0.$$

Proposition 1.2 (Convergences L^p implies in probability). Consider $p \in [1, \infty)$ and a sequence of random variables $X : \Omega \rightarrow \mathbb{R}^{\mathbb{N}}$ defined on a probability space (Ω, \mathcal{F}, P) such that $\lim_n X_n = X_\infty$ in L^p , then $\lim_n X_n = X_\infty$ in probability.

Proof. Let $\epsilon > 0$, then from the Markov's inequality applied to random variable $|X_n - X|_p^p$, we have

$$P \{|X_n - X_\infty| > \epsilon\} \leq \frac{\mathbb{E} |X_n - X_\infty|^p}{\epsilon}.$$

□

Example 1.3 (Convergence almost surely doesn't imply convergence in L^p). Consider the probability space $([0, 1], \mathcal{B}([0, 1]), \lambda)$ such that $\lambda([a, b]) = b - a$ for all $0 \leq a \leq b \leq 1$. We define the scaled indicator random variable $X_n : \Omega \rightarrow \{0, 1\}$ such that

$$X_n(\omega) = 2^n \mathbb{1}_{[0, \frac{1}{n}]}(\omega).$$

We define $N = \{0\}$, and for any $\omega \notin N$, we can find $m \triangleq \lceil \frac{1}{\omega} \rceil$, such that for all $n > m$, we have $X_n(\omega) = 0$. Since $\lambda(N) = 0$, it implies that $\lim_n X_n = 0$ a.s. However, we see that $\mathbb{E} |X_n|^p = \frac{2^{np}}{n}$.

Remark 2. Convergence almost surely implies convergence in probability. Therefore, above examples also serves as a counterexample to the fact that convergence almost surely doesn't imply convergence in L^p .

Theorem 1.4 (L^2 weak law of large numbers). Consider a sequence of uncorrelated random variables $X : \Omega \rightarrow \mathbb{R}^{\mathbb{N}}$ defined on a probability space (Ω, \mathcal{F}, P) such that $\mathbb{E} X_n = \mu$ and $\text{Var}(X_n) = \sigma^2$ for all $n \in \mathbb{N}$. Defining the sum $S_n \triangleq \sum_{i=1}^n X_i$ and the n -empirical mean $\bar{X}_n \triangleq \frac{S_n}{n}$, we have $\lim_n \bar{X}_n = \mu$ in L^2 and in probability.

Proof. From the uncorrelatedness of random sequence X , and linearity of expectation, we get

$$\text{Var}(\bar{X}_n) = \mathbb{E}(\bar{X}_n - \mu)^2 = \frac{1}{n^2} \mathbb{E}(S_n - n\mu)^2 = \frac{\sigma^2}{n}.$$

It follows that $\lim_n \bar{X}_n = \mu$ in L^2 . Since the convergence in L^p implies convergence in probability, the result holds. □

Theorem 1.5 (L^1 weak law of large numbers). Consider an i.i.d. random sequence $X : \Omega \rightarrow \mathbb{R}^{\mathbb{N}}$ defined on a probability space (Ω, \mathcal{F}, P) such that $\mathbb{E}|X_1| < \infty$ and $\mathbb{E}X_1 = \mu$. Defining the sum $S_n \triangleq \sum_{i=1}^n X_i$ and the n -empirical mean $\bar{X}_n \triangleq \frac{S_n}{n}$, we have $\lim_n \bar{X}_n = \mu$ in probability.

Example 1.6 (Convergence in L^p doesn't imply almost surely). Consider the probability space $([0,1], \mathcal{B}([0,1]), \lambda)$ such that $\lambda([a,b]) = b - a$ for all $0 \leq a \leq b \leq 1$. For each $k \in \mathbb{N}$, we consider the sequence $S_k = \sum_{i=1}^k i$, and define integer intervals $I_k \triangleq \{S_{k-1} + 1, \dots, S_k\}$. Clearly, the intervals $(I_k : k \in \mathbb{N})$ partition the natural numbers, and each $n \in \mathbb{N}$ lies in some I_{k_n} , such that $n = S_{k_n-1} + i_n$ for $i_n \in [k_n]$. Therefore, for each $n \in \mathbb{N}$, we define indicator random variable $X_n : \Omega \rightarrow \{0,1\}$ such that

$$X_n(\omega) \mathbb{1}_{\left[\frac{i_n-1}{k_n}, \frac{i_n}{k_n}\right]}(\omega).$$

For any $\omega \in [0,1]$, we have $X_n(\omega) = 1$ for infinitely many values since there exist infinitely many (i,k) pairs such that $\frac{(i-1)}{k} \leq \omega \leq \frac{i}{k}$, and hence $\limsup_n X_n(\omega) = 1$ and hence $\lim_n X_n(\omega) \neq 0$. However, $\lim_n X_n(\omega) = 0$ in L^p , since

$$\mathbb{E}|X_n|^p = \lambda\{X_n(\omega) \neq 0\} = \frac{1}{k_n}.$$

2 L^1 convergence theorems

Theorem 2.1 (Monotone Convergence Theorem). Consider a non-decreasing non-negative random sequence $X : \Omega \rightarrow \mathbb{R}_+^{\mathbb{N}}$ defined on a probability space (Ω, \mathcal{F}, P) , such that $X_n \in L^1$ for all $n \in \mathbb{N}$. Let $X_\infty(\omega) = \sup_n X_n(\omega)$ for all $\omega \in \Omega$, then $\mathbb{E}X_\infty = \sup_n \mathbb{E}X_n$.

Proof. From the monotonicity of sequence X and the monotonicity of expectation, we have $\sup_n \mathbb{E}X_n \leq \mathbb{E}X_\infty$. Let $\alpha \in (0,1)$ and $Y : \Omega \rightarrow \mathbb{R}_+$ a non-negative simple random variable such that $Y \leq X_\infty$. We define

$$E_n \triangleq \{\omega \in \Omega : X_n(\omega) \geq \alpha Y\} \in \mathcal{F}.$$

From the monotonicity of sequence X , the sequence of events $(E_n \in \mathcal{F} : n \in \mathbb{N})$ are monotonically non-decreasing such that $\cup_{n \in \mathbb{N}} E_n = \Omega$. It follows that

$$\alpha \mathbb{E}[Y \mathbb{1}_{E_n}] \leq \mathbb{E}[X_n \mathbb{1}_{E_n}] \leq \mathbb{E}X_n.$$

We will use the fact that $\lim_n \mathbb{E}[Y \mathbb{1}_{E_n}] = \mathbb{E}[Y]$, then $\alpha \mathbb{E}Y \leq \sup_n \mathbb{E}X_n$. Taking supremum over all $\alpha \in (0,1)$ and all simple functions $Y \leq X_\infty$, we get $\mathbb{E}X_\infty \leq \sup_n \mathbb{E}X_n$. \square

Theorem 2.2 (Fatou's Lemma). Consider a non-negative random sequence $X : \Omega \rightarrow \mathbb{R}_+^{\mathbb{N}}$ defined on a probability space (Ω, \mathcal{F}, P) . Let $X_\infty(\omega) \triangleq \liminf_n X_n(\omega)$ for all $\omega \in \Omega$, then $\mathbb{E}X_\infty \leq \liminf_n \mathbb{E}X_n$.

Proof. We define $Y_n \triangleq \inf_{k \geq n} X_k$ for all $n \in \mathbb{N}$. It follows that $Y : \Omega \rightarrow \mathbb{R}_+^{\mathbb{N}}$ is a non-negative non-decreasing sequence of random variables, and $X_\infty = \sup_n Y_n = \lim_n Y_n$. By Monotone convergence theorem applies to Y , we have $\mathbb{E}X_\infty = \sup_n \mathbb{E}Y_n$. The result follows from the monotonicity of expectation, and the fact that $Y_n \leq X_k$ for all $k \geq n$, to get $\mathbb{E}Y_n \leq \inf_{k \geq n} \mathbb{E}X_k$. \square

Theorem 2.3 (Dominated Convergence Theorem). Let $X : \Omega \rightarrow \mathbb{R}^{\mathbb{N}}$ be a random sequence defined on a probability space (Ω, \mathcal{F}, P) . If $\lim_n X_n = X_\infty$ a.s. and there exists a $Y : \Omega \rightarrow \mathbb{R}_+$ such that $Y \in L^1$ and $|X_n| \leq Y$ a.s., then $\mathbb{E}X_\infty = \lim_n \mathbb{E}X_n$.

Proof. From the hypothesis, we have $Y + X_n \geq 0$ a.s. and $Y - X_n \geq 0$ a.s. Therefore, from Fatou's Lemma and linearity of expectation, we have

$$\mathbb{E}Y + \mathbb{E}X_\infty \leq \liminf_n \mathbb{E}(Y + X_n) = \mathbb{E}Y + \liminf_n \mathbb{E}X_n, \quad \mathbb{E}Y - \mathbb{E}X_\infty \leq \liminf_n \mathbb{E}(Y - X_n) = \mathbb{E}Y - \limsup_n \mathbb{E}X_n.$$

Therefore, we have $\limsup_n \mathbb{E}X_n \leq \mathbb{E}X_\infty \leq \liminf_n \mathbb{E}X_n$, and the result follows. \square

3 Uniform integrability

Definition 3.1 (uniform integrability). A family $(X_t \in L^1 : t \in T)$ of random variables indexed by T is **uniformly integrable** if

$$\lim_{a \rightarrow \infty} \sup_{t \in T} \mathbb{E}[|X_t| \mathbb{1}_{\{|X_t| > a\}}] = 0.$$

Example 3.2 (Single element family). If $|T| = 1$, then the family is uniformly integrable, since $X_1 \in L^1$ and $\lim_a \mathbb{E}[|X_1| \mathbb{1}_{\{|X_1| > a\}}] = 0$. This is due to the fact that $(X_n \triangleq |X| \mathbb{1}_{\{|X| \leq n\}} : n \in \mathbb{N})$ is a sequence of increasing random variables $\lim_n X_n = X$. From monotone convergence theorem, we get $\lim_n \mathbb{E}|X_n| = \mathbb{E} \lim_n |X_n|$. Therefore,

$$\lim_a \mathbb{E}[|X| \mathbb{1}_{\{|X| > a\}}] = \mathbb{E}|X| - \lim_a \mathbb{E}[|X| \mathbb{1}_{\{|X| \leq a\}}] = 0.$$

Proposition 3.3. Let $X \in L^p$ and $(A_n : n \in \mathbb{N}) \subset \mathcal{F}$ be a sequence of events such that $\lim_n P(A_n) = 0$, then

$$\lim_n \||X| \mathbb{1}_{A_n}\|_p = 0.$$

Example 3.4 (Dominated family). If there exists $Y \in L^1$ such that $\sup_{t \in T} |X_t| \leq |Y|$, then the family of random variables $(X_t : t \in T)$ is uniformly integrable. This is due to the fact that

$$\sup_{t \in T} \mathbb{E}[|X_t| \mathbb{1}_{\{|X_t| > a\}}] \leq \mathbb{E}[|Y| \mathbb{1}_{\{|Y| > a\}}].$$

Example 3.5 (Finite family). then the family of random variables $(X_t : t \in T)$ is uniformly integrable. This is due to the fact that $\sup_{t \in T} |X_t| \leq \sum_{t \in T} |X_t| \in L^1$.

Theorem 3.6 (Convergence in probability with uniform integrability implies convergence in L^p). Consider a sequence of random variables $(X_n : n \in \mathbb{N}) \subset L^p$ for $p \geq 1$. Then the following are equivalent.

- (a) The sequence $(X_n : n \in \mathbb{N})$ converges in L^p , i.e. $\lim_n \mathbb{E}|X_n - X|^p = 0$.
- (b) The sequence $(X_n : n \in \mathbb{N})$ is Cauchy in L^p , i.e. $\lim_{m, n \rightarrow \infty} \mathbb{E}|X_n - X_m|^p = 0$.
- (c) $\lim_n X_n = X$ in probability and the sequence $(|X_n|^p : n \in \mathbb{N})$ is uniformly integrable.

Proof. For a random sequence $(X_n : n \in \mathbb{N})$ in L^p , we will show that (a) \implies (b) \implies (c) \implies (a).

(a) \implies (b): We assume the sequence $(X_n : n \in \mathbb{N})$ converges in L^p . Then, from Minkowski's inequality, we can write

$$(\mathbb{E}|X_n - X_m|^p)^{\frac{1}{p}} \leq (\mathbb{E}|X_n - X|^p)^{\frac{1}{p}} + (\mathbb{E}|X_m - X|^p)^{\frac{1}{p}}.$$

(b) \implies (c): We assume that the sequence $(X_n : n \in \mathbb{N})$ is Cauchy in L^p , i.e. $\lim_{m, n \rightarrow \infty} \mathbb{E}|X_n - X_m|^p = 0$. Let $\epsilon > 0$, then for each $n \in \mathbb{N}$, there exists N_ϵ such that for all $n, m \geq N_\epsilon$

$$\mathbb{E}|X_n - X_m|^p \leq \frac{\epsilon}{2}.$$

Let $A_a = \{\omega \in A : |X_n| > a\}$. Then, using triangle inequality and the fact that $\mathbb{1}_{A_a} \leq 1$, from the linearity and monotonicity of expectation, we can write for $n \geq N_\epsilon$

$$(\mathbb{E}[|X_n|^p \mathbb{1}_{\{|X_n|>a\}}])^{\frac{1}{p}} \leq (\mathbb{E}[|X_{N_\epsilon}|^p \mathbb{1}_{A_a}])^{\frac{1}{p}} + (\mathbb{E}[|X_n - X_{N_\epsilon}|^p])^{\frac{1}{p}} \leq (\mathbb{E}[|X_{N_\epsilon}|^p \mathbb{1}_{A_a}])^{\frac{1}{p}} + \frac{\epsilon}{2}.$$

Therefore, we can write $\sup_n \mathbb{E}[|X_n|^p \mathbb{1}_{\{|X_n|>a\}}] \leq \sup_{m \leq N_\epsilon} \mathbb{E}[|X_m|^p \mathbb{1}_{A_a}] + \frac{\epsilon}{2}$. Since $(|X_n|^p : n \leq N_\epsilon)$ is finite family of random variables in L^1 , it is uniformly integrable. Therefore, there exists $a_\epsilon \in \mathbb{R}_+$ such that $\sup_{m \leq N_\epsilon} (\mathbb{E}[|X_m|^p \mathbb{1}_{A_{a_\epsilon}}])^{\frac{1}{p}} < \frac{\epsilon}{2}$. Taking $a' = \max\{a, a_\epsilon\}$, we get $\sup_n (\mathbb{E}[|X_n|^p \mathbb{1}_{\{|X_n|>a'\}}])^{\frac{1}{p}} \leq \epsilon$. Since the choice of ϵ was arbitrary, it follows that

$$\lim_{a \rightarrow \infty} \sup_n (\mathbb{E}[|X_n|^p \mathbb{1}_{\{|X_n|>a'\}}])^{\frac{1}{p}} = 0.$$

The convergence in probability follows from the Markov inequality, i.e.

$$P\{|X_n - X_m|^p > \epsilon\} \leq \frac{1}{\epsilon} \mathbb{E}|X_n - X_m|^p.$$

(c) \implies (a) : Since the sequence $(X_n : n \in \mathbb{N})$ is convergent in probability to a random variable X , there exists a subsequence $(n_k : k \in \mathbb{N}) \subset \mathbb{N}$ such that $\lim_k X_{n_k} = X$ a.s. Since $(|X_n|^p : n \in \mathbb{N})$ is a family of uniformly integrable sequence, by Fatou's Lemma

$$\mathbb{E}|X|^p \leq \liminf_k \mathbb{E}|X_{n_k}|^p \leq \sup_n \mathbb{E}|X_n|^p < \infty.$$

Therefore, $X \in L^1$, and we define $A_n(\epsilon) = \{|X_n - X| > \epsilon\}$ for any $\epsilon > 0$. From Minkowski's inequality, we get

$$\|X_n - X\|_p \leq \left\| (X_n - X) \mathbb{1}_{\{|X_n - X|^p \leq \epsilon\}} \right\|_p + \left\| X_n \mathbb{1}_{A_n(\epsilon)} \right\|_p + \left\| X \mathbb{1}_{A_n(\epsilon)} \right\|_p.$$

We can check that $\left\| (X_n - X) \mathbb{1}_{A_n^c(\epsilon)} \right\|_p \leq \epsilon$. Further, since $\lim_n X_n = X$ in probability, $(A_n : n \in \mathbb{N}) \subset \mathcal{F}$ is decreasing sequence of events, and since $X_n, X \in L^1$, we have $\lim_n \left\| X_n \mathbb{1}_{A_n(\epsilon)} \right\|_p = \lim_n \left\| X \mathbb{1}_{A_n(\epsilon)} \right\|_p = 0$.

□