

# Lecture-14: Weak convergence

## 1 Convergence in distribution

**Definition 1.1 (convergence in distribution).** A random sequence  $X : \Omega \rightarrow \mathbb{R}^{\mathbb{N}}$  defined on a probability space  $(\Omega, \mathcal{F}, P)$  **converges in distribution** to a random variable  $X_{\infty} : \Omega' \rightarrow \mathbb{R}$  defined on a probability space  $(\Omega', \mathcal{F}', P')$  if

$$\lim_n F_{X_n}(x) = F_{X_{\infty}}(x)$$

at all continuity points  $x$  of  $F_{X_{\infty}}$ . Convergence in distribution is denoted by  $\lim_n X_n = X_{\infty}$  in distribution.

**Proposition 1.2.** Consider a random sequence  $X : \Omega \rightarrow \mathbb{R}^{\mathbb{N}}$  defined on a probability space  $(\Omega, \mathcal{F}, P)$  and a random variable  $X_{\infty} : \Omega' \rightarrow \mathbb{R}$  defined on another probability space  $(\Omega', \mathcal{F}', P')$ . Then the following statements are equivalent.

- (a)  $\lim_n X_n = X_{\infty}$  in distribution.
- (b)  $\lim_n \mathbb{E}[g(X_n)] = \mathbb{E}[g(X_{\infty})]$  for any bounded continuous function  $g$ .
- (c) Characteristic functions converge point-wise, i.e.  $\lim_n \Phi_{X_n}(u) = \Phi_{X_{\infty}}(u)$  for each  $u \in \mathbb{R}$ .

*Proof.* Let  $X : \Omega \rightarrow \mathbb{R}^{\mathbb{N}}$  be a sequence of random variables and let  $X_{\infty} : \Omega' \rightarrow \mathbb{R}$  be a random variable. We will show that (a)  $\implies$  (b)  $\implies$  (c)  $\implies$  (a).

- (a)  $\implies$  (b): Let  $\lim_n X_n = X_{\infty}$  in distribution, then  $\lim_n \int_{x \in \mathbb{R}} g(x) dF_{X_n}(x) = \int_{x \in \mathbb{R}} g(x) \lim_n dF_{X_n}(x)$  by the bounded convergence theorem for any bounded continuous function  $g$ .
- (b)  $\implies$  (c): Let  $\lim_n \mathbb{E}[g(X_n)] = \mathbb{E}[g(X_{\infty})]$  for any bounded continuous function  $g$ . Taking  $g(x) = e^{jux}$ , we get the result.
- (c)  $\implies$  (a): The proof of this part is technical and is omitted.

□

**Example 1.3 (Convergence in distribution but not in probability).** Consider a sequence of non-degenerate continuous *i.i.d.* random variables  $X : \Omega \rightarrow \mathbb{R}^{\mathbb{N}}$  and independent random variable  $Y : \Omega' \rightarrow \mathbb{R}$  with the common distribution  $F_Y$ . Then  $F_{X_n} = F_Y$  for all  $n \in \mathbb{N}$ , and hence  $\lim_n X_n = Y$  in distribution. When each  $X_n$  is zero mean Gaussian with variance  $\sigma^2$ , it follows that  $X_n - Y$  is zero mean Gaussian with variance  $2\sigma^2$ . Therefore, for  $\epsilon < \sigma\sqrt{\pi}$  and all  $n \in \mathbb{N}$

$$P\{|X_n - Y| \leq \epsilon\} = \frac{1}{\sqrt{4\pi\sigma^2}} \int_{z \in [-\epsilon, \epsilon]} e^{-z^2/4\sigma^2} dz \leq \frac{\epsilon}{\sigma\sqrt{\pi}} < 1.$$

**Lemma 1.4 (Convergence in probability implies in distribution).** Consider a sequence  $X : \Omega \rightarrow \mathbb{R}^{\mathbb{N}}$  of random variables and a random variable  $X_{\infty} : \Omega \rightarrow \mathbb{R}$  defined on a probability space  $(\Omega, \mathcal{F}, P)$ , such that  $\lim_n X_n = X_{\infty}$  in probability, then  $\lim_n X_n = X_{\infty}$  in distribution.

*Proof.* Fix  $\epsilon > 0$ , and consider the event

$$E_n(\epsilon) \triangleq \{\omega \in \Omega : |X_n(\omega) - X_{\infty}(\omega)| > \epsilon\} = \{X_n \notin [X_{\infty} - \epsilon, X_{\infty} + \epsilon]\} \in \mathcal{F}.$$

We further define events  $A_{X_n}(x) \triangleq \{X_n \leq x\}$  and  $A_{X_\infty}(x) \triangleq \{X_\infty \leq x\}$ , then we can write

$$\begin{aligned} A_{X_n}(x) \cap A_{X_\infty}(x + \epsilon) &\subseteq A_{X_\infty}(x + \epsilon), & A_{X_n}(x) \cap A_{X_\infty}^c(x + \epsilon) &\subseteq A_n(\epsilon), \\ A_{X_\infty}(x - \epsilon) \cap A_{X_n}(x) &\subseteq A_{X_n}(x), & A_{X_\infty}(x - \epsilon) \cap A_{X_n}^c(x) &\subseteq A_n(\epsilon). \end{aligned}$$

From the above set relations, law of total probability, and union bound, we have

$$F_{X_\infty}(x - \epsilon) - P(E_n) \leq F_{X_n}(x) \leq F_{X_\infty}(x + \epsilon) + P(E_n).$$

From the convergence in probability, we have  $\lim_n P(E_n) = 0$ . Therefore, we get

$$F_{X_\infty}(x - \epsilon) \leq \liminf_n F_{X_n}(x) \leq \limsup_n F_{X_n}(x) \leq F_{X_\infty}(x + \epsilon).$$

We get the result at the continuity points of  $F_X$ , since the choice of  $\epsilon$  was arbitrary.  $\square$

**Theorem 1.5 (Central Limit Theorem).** Consider an i.i.d. random sequence  $X : \Omega \rightarrow \mathbb{R}^{\mathbb{N}}$  defined on a probability space  $(\Omega, \mathcal{F}, P)$ , with  $\mathbb{E}X_n = \mu$  and  $\text{Var}(X_n) = \sigma^2$  for all  $n \in \mathbb{N}$ . We define the  $n$ -sum as  $S_n \triangleq \sum_{i=1}^n X_i$  and consider a standard normal random variable  $Y : \Omega \rightarrow \mathbb{R}$  with density function  $f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$  for all  $y \in \mathbb{R}$ . Then,

$$\lim_n \frac{S_n - n\mu}{\sigma\sqrt{n}} = Y \text{ in distribution.}$$

*Proof.* The classical proof is using the characteristic functions. Let  $Z_i \triangleq \frac{X_i - \mu}{\sigma}$  for all  $i \in \mathbb{N}$ , then the shifted and scaled  $n$ -sum is given by  $\frac{S_n - n\mu}{\sigma\sqrt{n}} = \frac{1}{\sqrt{n}} \sum_{i=1}^n Z_i$ . We use the third equivalence in Proposition 1.2 to show that the characteristic function of converges to the characteristic function of the standard normal. We define the characteristic functions

$$\Phi_n(u) \triangleq \mathbb{E} \exp\left(ju \frac{(S_n - n\mu)}{\sigma\sqrt{n}}\right), \quad \Phi_{Z_i}(u) \triangleq \mathbb{E} \exp(juZ_i), \quad \Phi_Y(u) \triangleq \mathbb{E} \exp(juY).$$

We can compute the characteristic function of the standard normal as

$$\Phi_Y(u) = \frac{1}{\sqrt{2\pi}} \int_{y \in \mathbb{R}} e^{-\frac{y^2}{2}} \exp\left(-\frac{(y - ju)^2}{2}\right) dy = e^{-\frac{u^2}{2}}.$$

Since the random sequence  $Z : \Omega \rightarrow \mathbb{R}^{\mathbb{N}}$  is a zero mean i.i.d. sequence, using the Taylor expansion of the characteristic function, we have

$$\Phi_n(u) = \prod_{i=1}^n \mathbb{E} \exp\left(ju \frac{(X_i - \mu)}{\sigma\sqrt{n}}\right) = \left[\Phi_{Z_1}\left(\frac{u}{\sqrt{n}}\right)\right]^n = \left[1 - \frac{u^2}{2n} + o\left(\frac{u^2}{n}\right)\right]^n.$$

For any  $u \in \mathbb{R}$ , taking limit  $n \in \mathbb{N}$ , we get the result.  $\square$

## 2 Strong law of large numbers

**Definition 2.1.** For a random sequence  $X : \Omega \rightarrow \mathbb{R}^{\mathbb{N}}$  defined on a probability space  $(\Omega, \mathcal{F}, P)$  with bounded mean  $\mathbb{E}|X_n| < \infty$  for all  $n \in \mathbb{N}$ , we define the  $n$ -sum as  $S_n \triangleq \sum_{i=1}^n X_i$  and the empirical  $n$ -mean  $\frac{S_n}{n}$  for each  $n \in \mathbb{N}$ . For each  $n \in \mathbb{N}$ , we define event

$$E_n \triangleq \{\omega \in \Omega : |S_n - \mathbb{E}S_n| > n\epsilon\} \in \mathcal{F}.$$

**Theorem 2.2 ( $L^4$  strong law of large numbers).** Let  $X : \Omega \rightarrow \mathbb{R}^{\mathbb{N}}$  be a sequence of independent random variables defined on probability space  $(\Omega, \mathcal{F}, P)$  with bounded mean  $\mathbb{E}X_n$  for each  $n \in \mathbb{N}$  and uniformly bounded fourth central moment  $\sup_{n \in \mathbb{N}} \mathbb{E}(X_n - \mathbb{E}X_n)^4 \leq B < \infty$ . Then, the empirical  $n$ -mean converges to  $\lim_n \frac{\mathbb{E}S_n}{n}$  almost surely.

*Proof.* Recall that  $\mathbb{E}(S_n - \mathbb{E}S_n)^4 = \mathbb{E}(\sum_{i=1}^n (X_i - \mathbb{E}X_i))^4 = \sum_{i=1}^n \mathbb{E}(X_i - \mathbb{E}X_i)^4 + 3 \sum_{i=1}^n \sum_{j \neq i} \mathbb{E}(X_i - \mathbb{E}X_i)^2 \mathbb{E}(X_j - \mathbb{E}X_j)^2$ . Recall that when the fourth moment is bounded, then so is second moment. Hence,  $\sup_{i \in \mathbb{N}} \mathbb{E}(X_i - \mathbb{E}X_i)^2 \leq C$  for some  $C \in \mathbb{R}_+$ . Therefore, from the Markov's inequality, we have

$$P(E_n) \leq \frac{\mathbb{E}(S_n - \mathbb{E}S_n)^4}{n^4 \epsilon^4} \leq \frac{nB + 3n(n-1)C}{n^4 \epsilon^4}.$$

It follows that the  $\sum_{n \in \mathbb{N}} P(E_n) < \infty$ , and hence by Borel Canteli Lemma, we have

$$P\{E_n^c \text{ for all but finitely many } n\} = 1.$$

Since, the choice of  $\epsilon$  was arbitrary, the result follows.  $\square$

**Theorem 2.3 ( $L^2$  strong law of large numbers).** Let  $X : \Omega \rightarrow \mathbb{R}^{\mathbb{N}}$  be a sequence of pair-wise uncorrelated random variables defined on a probability space  $(\Omega, \mathcal{F}, P)$  with bounded mean  $\mathbb{E}X_n$  for all  $n \in \mathbb{N}$  and uniformly bounded variance  $\sup_{n \in \mathbb{N}} \text{Var}(X_n) \leq B < \infty$ . Then, the empirical  $n$ -mean converges to  $\lim_n \frac{\mathbb{E}S_n}{n}$  almost surely.

*Proof.* For each  $n \in \mathbb{N}$ , we define events  $F_n \triangleq E_{n^2}$ , and

$$G_n \triangleq \left\{ \max_{n^2 \leq k < (n+1)^2} |S_k - S_{n^2} - \mathbb{E}(S_k - S_{n^2})| > n^2 \epsilon \right\} = \bigcup_{n^2 \leq k < (n+1)^2} \left\{ \omega \in \Omega : \left| \sum_{i=n^2}^k (X_i - \mathbb{E}X_i) \right| > n^2 \epsilon \right\}.$$

From the Markov's inequality and union bound, we have

$$P(F_n) \leq \frac{\sum_{i=1}^{n^2} \text{Var}(X_i)}{n^4 \epsilon^2} \leq \frac{B}{n^2 \epsilon^2}, \quad P(G_n) \leq \sum_{k=n^2}^{(n+1)^2-1} \frac{(k - n^2 + 1)B}{n^4 \epsilon^2} \leq \frac{(2n+1)^2 B}{n^4 \epsilon^2}.$$

Therefore,  $\sum_{n \in \mathbb{N}} P(F_n) < \infty$  and  $\sum_{n \in \mathbb{N}} P(G_n) < \infty$ , and hence by Borel Canteli Lemma, we have

$$\lim_n \frac{S_{n^2} - \mathbb{E}S_{n^2}}{n^2} = \lim_n \frac{S_k - S_{n^2} - \mathbb{E}(S_k - S_{n^2})}{n^2} = 0 \text{ a.s.}$$

The result follows from the fact that for any  $k \in \mathbb{N}$ , there exists  $n \in \mathbb{N}$  such that  $k \in \{n^2, \dots, (n+1)^2 - 1\}$  and hence

$$\frac{|S_k - \mathbb{E}S_k|}{k} \leq \left( \frac{|S_{n^2} - \mathbb{E}S_{n^2}|}{n^2} + \frac{|S_k - S_{n^2} - \mathbb{E}(S_k - S_{n^2})|}{n^2} \right).$$

$\square$

**Theorem 2.4 ( $L^1$  strong law of large numbers).** Let  $X : \Omega \rightarrow \mathbb{R}^{\mathbb{N}}$  be a random sequence defined on a probability space  $(\Omega, \mathcal{F}, P)$  such that  $\sup_{n \in \mathbb{N}} \mathbb{E}|X_n| \leq B < \infty$ . Then, the empirical  $n$ -mean converges to  $\lim_n \frac{\mathbb{E}S_n}{n}$  almost surely.