Lecture-18: Stopping Times

1 Stopping times

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Consider a random process $X : \Omega \to \mathcal{X}^T$ defined on this probability space with state space $\mathcal{X} \subseteq \mathbb{R}$ and ordered index set $T \subseteq \mathbb{R}$ considered as time.

Definition 1.1. A collection of event spaces denoted $\mathcal{F}_\bullet = (\mathcal{F}_t \subseteq \mathcal{F} : t \in T)$ is called a filtration if $\mathcal{F}_s \subseteq \mathcal{F}_t$ for all $s \leqslant t$.

Remark 1. For the random process $X : \Omega \to \mathcal{X}^T$, we can find the event space generated by all random variables until time t as $\mathcal{G}_t \triangleq \sigma(X_s, s \leqslant t)$. The collection of event spaces $\mathcal{G}_\bullet = (\mathcal{G}_t : t \in T)$ is a filtration.

Definition 1.2. The natural filtration associated with a random process $X : \Omega \to \mathcal{X}^T$ is given by $\mathcal{F}_\bullet = (\mathcal{F}_t : t \in T)$ where $\mathcal{F}_t \triangleq \sigma(X_s, s \leqslant t)$.

Remark 2. For a random sequence $X : \Omega \to \mathcal{X}^{\mathbb{N}}$, the natural filtration is a sequence $\mathcal{F}_\bullet = (\mathcal{F}_n : n \in \mathbb{N})$ of event spaces $\mathcal{F}_n \triangleq \sigma(X_1, \dots, X_n)$ for all $n \in \mathbb{N}$.

Remark 3. If the random sequence X is independent, then the random sequence $(X_{n+j} : j \in \mathbb{N})$ is independent of the event space $\sigma(X_1, \dots, X_n)$.

Example 1.3. For a random walk S with step size sequence X, the natural filtration of the random walk is identical to that of the step size sequence. That is, $\sigma(S_1, \dots, S_n) = \sigma(X_1, \dots, X_n)$ for all $n \in \mathbb{N}$. This follows from the fact that for all $n \in \mathbb{N}$, we can can write $S_j = \sum_{i=1}^{j} X_i$ and $X_j = S_j - S_{j-1}$ for all $j \in [n]$. That is, there is a bijection between $(X_1, \dots, X_n)$ and $(S_1, \dots, S_n)$.

Definition 1.4. A random variable $\tau : \Omega \to T$ is called a **stopping time** with respect to a filtration $\mathcal{F}_\bullet$ if

(a) the event $\tau^{-1}(-\infty, t] \in \mathcal{F}_t$ for all $t \in T$, and

(b) the random variable τ is finite almost surely, i.e. $P\{\tau < \infty\} = 1$.

Remark 4. Intuitively, if we observe the process X sequentially, then the event $\{\tau \leqslant t\}$ can be completely determined by the observation $(X_s, s \leqslant t)$ until time t. The intuition behind a stopping time is that its realization is determined by the past and present events but not by future events. That is, given the history of the process until time t, we can tell whether the stopping time is less than or equal to t or not. In particular, $\mathbb{E}[\mathbb{1}_{\{\tau \leqslant t\}} \mid \mathcal{F}_t] = \mathbb{1}_{\{\tau \leqslant t\}}$ is either one or zero.

Definition 1.5 (First hitting time). For a process $X : \Omega \to \mathcal{X}^T$ and any Borel measurable set $A \in \mathcal{B}(\mathcal{X})$, we define the **first hitting time** $\tau_X^A : \Omega \to T \cup \{\infty\}$ for the process X to hit states in A, as $\tau_X^A \triangleq \inf\{t \in T : X_t \in A\}$.

Example 1.6. We observe that the event $\{\tau_X^A \leqslant t\} = \{X_s \in A \text{ for some } s \leqslant t\} \in \mathcal{F}_t$ for all $t \in T$. It follows that, if τ_A is finite almost surely, then τ_A is a stopping time with respect to filtration $\mathcal{F}_\bullet$.

Theorem 1.7. *For a random sequence $X : \Omega \to \mathcal{X}^{\mathbb{N}}$, an almost sure finite discrete random variable $\tau : \Omega \to \mathbb{N} \cup \{\infty\}$ is a **stopping time** with respect to this random sequence X iff the event $\{\tau = n\} \in \sigma(X_1, \dots, X_n)$ for all $n \in \mathbb{N}$.*

Proof. From Definition 1.4, we have $\{\tau = n\} = \{\tau \leqslant n\} \setminus \{\tau \leqslant n-1\} \in \sigma(X_1, \dots, X_n)$. Conversely, from the theorem hypothesis, it follows that $\{\tau \leqslant n\} = \cup_{m=1}^{n} \{\tau = m\} \in \sigma(X_1, \dots, X_n)$. □

Example 1.8. Consider a random sequence $X : \Omega \to \mathcal{X}^{\mathbb{N}}$, with the natural filtration $\mathcal{F}_\bullet$, and a measurable set $A \in \mathcal{B}(\mathcal{X})$. If the first hitting time $\tau_X^A : \Omega \to \mathbb{N} \cup \{\infty\}$ for the sequence X to hit set A is almost surely finite, then τ_X^A is a hitting time. This follows from the fact that $\{\tau_X^A = n\} = \cap_{k=1}^{n-1} \{X_k \notin A\} \cap \{X_n \in A\} \in \mathcal{F}_n$ for each $n \in \mathbb{N}$.

Definition 1.9. Consider a random process $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ with discrete state space $\mathcal{X} \subseteq \mathbb{R}$. For each state $y \in \mathcal{X}$, we define $\tau_X^{\{y\},0} \triangleq 0$ and inductively define the k**th hitting time to a state** y after time $t = 0$, as

$$\tau_X^{\{y\},k} \triangleq \inf\left\{t > \tau_X^{(\{y\},k-1} : X_t = y\right\}, \quad k \in \mathbb{N}.$$

Remark 5. We observe that $\left\{\tau_X^{\{y\},k} \leqslant t\right\} \in \mathcal{F}_t$ for all times $t \in \mathbb{R}_+$. Hence if $\tau_X^{\{y\},k}$ is almost surely finite, then it is a stopping time for the process X.

Definition 1.10. For a discrete valued random sequence $X : \Omega \to \mathcal{X}^{\mathbb{N}}$, the number of visits to a state $y \in \mathcal{X}$ in first n time steps is defined as $N_y(n) \triangleq \sum_{k=1}^n \mathbb{1}_{\{X_k = y\}}$ for all $n \in \mathbb{N}$.

Remark 6. We observe that $N_y : \Omega \to \mathbb{Z}_+^{\mathbb{Z}_+}$ is a random walk with the Bernoulli step size sequence $(\mathbb{1}_{\{X_k=y\}} : k \in \mathbb{N})$. Further, $\tau_X^{\{y\},k} = \tau_{N_y}^{\{k\}} = \inf\{n \in \mathbb{N} : N_y(n) = k\}$. We also observe that $\{N_y(n) \leqslant k\} = \{\tau_y^{(k+1)} > n\}$ and $\{N_y(n) = k\} = \{\tau_y^k \leqslant n < \tau_y^{(k+1)}\}$.

Remark 7. We observe that number of visits to state y in first n steps of X is also given by

$$N_y(n) = \sup\left\{k \in \mathbb{Z}_+ : \tau_X^{\{y\},k} \leqslant n\right\} = \inf\left\{k \in \mathbb{N} : \tau_X^{\{y\},k} > n\right\} - 1 = \sum_{k \in \mathbb{N}} \mathbb{1}_{\left\{\tau_X^{\{y\},k} \leqslant n\right\}}.$$

This implies that $N_y(n) + 1$ is the first hitting time to state $(n+1)$ for the increasing random sequence $(\tau_X^{\{y\},k} : k \in \mathbb{N})$.

Lemma 1.11 (Wald's Lemma). *Consider a random walk $S : \Omega \to \mathbb{R}^{\mathbb{Z}_+}$ with* i.i.d. *step-sizes $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ having finite $\mathbb{E}|X_1|$. Let τ be a finite mean stopping time with respect to this random walk. Then,*

$$\mathbb{E}[S_\tau] = \mathbb{E}[X_1]\,\mathbb{E}[\tau].$$

Proof. Recall that the event space generate by the random walk and the step-sizes are identical. From the independence of step sizes, it follows that X_n is independent of $\sigma(X_0, X_1, \dots, X_{n-1})$. Since τ is a stopping time with respect to random walk S, we observe that $\{\tau \geqslant n\} = \{\tau > n-1\} \in \sigma(X_0, X_1, \dots, X_{n-1})$, and hence it follows that random variable X_n and indicator $\mathbb{1}_{\{\tau \geqslant n\}}$ are independent and $\mathbb{E}[X_n \mathbb{1}_{\{\tau \geqslant n\}}] = \mathbb{E}X_1 \mathbb{E}\mathbb{1}_{\{\tau \geqslant n\}}$. Therefore,

$$\mathbb{E}[S_\tau] = \mathbb{E}[\sum_{n=1}^{\tau} X_n] = \mathbb{E}[\sum_{n \in \mathbb{N}} X_n \mathbb{1}_{\{\tau \geqslant n\}}] = \sum_{n \in \mathbb{N}} \mathbb{E}X_1 \mathbb{E}\left[\mathbb{1}_{\{\tau \geqslant n\}}\right] = \mathbb{E}X_1 \mathbb{E}\left[\sum_{n \in \mathbb{N}} \mathbb{1}_{\{\tau \geqslant n\}}\right] = \mathbb{E}[X_1]\mathbb{E}[\tau].$$

We exchanged limit and expectation in the above step, which is not always allowed. We were able to do it by the application of dominated convergence theorem. ☐

Corollary 1.12. *Consider the stopping time $\tau_S^{\{i\}} \triangleq \min\{n \in \mathbb{N} : S_n = i\}$ for an integer random walk $S : \Omega \to \mathbb{Z}_+^{\mathbb{Z}_+}$ with* i.i.d. *step size sequence $X : \Omega \to \mathbb{Z}^{\mathbb{N}}$. Then, the mean of stopping time $\mathbb{E}\tau_S^{\{i\}} = i/\mathbb{E}X_1$.*

Proof. This follows from the Wald's Lemma and the fact that $S_{\tau_i} = i$. ☐

1.1 Properties of stopping time

Lemma 1.13. *Let τ_1, τ_2 be two stopping times with respect to filtration $\mathcal{F}_\bullet$. Then the following hold true.*

*i_* $\min\{\tau_1, \tau_2\}$ *is a stopping time.*

ii_ If T is separable, then $\tau_1 + \tau_2$ is a stopping time.

Proof. Let τ_1, τ_2 be stopping times with respect to a filtration $\mathcal{F}_\bullet = (\mathcal{F}_t : t \in T)$.

i_ Result follows since the event $\{\min\{\tau_1, \tau_2\} > t\} = \{\tau_1 > t\} \cap \{\tau_2 > t\} \in \mathcal{F}_t$.

ii_ A topological space is called if it contains a countable dense set. Since $\mathbb{R}_+$ is separable and ordered, we assume $T = \mathbb{R}_+$ without any loss of generality. It suffices to show that the event $\{\tau_1 + \tau_2 \leqslant t\} \in \mathcal{F}_t$ for $T = \mathbb{R}_+$. To this end, we observe that $\{\tau_1 + \tau_2 \leqslant t\} = \bigcup_{s \in \mathbb{Q}_+ : s \leqslant t} \{\tau_1 \leqslant t - s, \tau_2 \leqslant s\} \in \mathcal{F}_t$.

□

Theorem 1.14 (Strong independence property). *Let $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ be an independent random process with natural filtration $\mathcal{F}_\bullet$, and $\tau : \Omega \to \mathbb{R}_+$ a stopping time with respect to $\mathcal{F}_\bullet$, then $(X_{\tau+s} : s \in \mathbb{R}_+)$ is independent of history $(X_s : s \leqslant \tau)$.*

Definition 1.15. We can define the k**th return time to state** y for the random process $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ as the interval between two successive visits to state y, that is for all $k \in \mathbb{N}$

$$H_X^{\{y\},k} \triangleq \tau_X^{\{y\},k} - \tau_X^{\{y\},k-1} = \inf\left\{ s \in \mathbb{R}_+ : X_{\tau_X^{\{y\},k-1}+s} = y \right\}.$$

Remark 8. We observe that $\tau_X^{\{y\},k)} = \sum_{j=1}^{k} H_X^{\{y\},j}$. Therefore, if $H_X^{\{y\},j}$ is almost sure finite for all $j \in [k]$, then the finite sum $\tau_X^{\{y\},k}$ is almost sure finite.

Remark 9. From the bijection between hitting and return times, we observe that $\sigma(\tau_X^{\{y\},k)} : k \in [n]) = \sigma(H_X^{\{y\},j} : j \in [n])$. If passage times are independent and *i.i.d.* from second passage time, then it follows from the Wald's Lemma that

$$\mathbb{E}H_X^{\{y\},1} + \mathbb{E}[N_y(n)]\mathbb{E}H_X^{\{y\},2} = (n+1)$$

Example 1.16. We also observe that the kth hitting time to $\{1\}$ by a Bernoulli step size sequence $X : \Omega \to \{0,1\}^{\mathbb{N}}$ is the first hitting time to $\{k\}$ by random walk $S : \Omega \to \mathbb{Z}_+^{\mathbb{Z}_+}$. That is,

$$\tau_X^{\{1\},k} = \tau_S^{\{k\}} = \inf\{n \in \mathbb{N} : S_n = k\} = \tau_S^{\{k-1\}} + \inf\left\{ n \in \mathbb{N} : S_{\tau_S^{\{k-1\}}+n} - S_{\tau_S^{\{k-1\}}} = 1 \right\}.$$

Lemma 1.17. *For an* i.i.d. *Bernoulli random sequence $X : \Omega \to \{0,1\}^{\mathbb{N}}$ where $\mathbb{E}X_1 \in (0,1)$, then the kth hitting time to state* 1 *is a stopping time, and $\tau_X^{\{1\},k} = \sum_{i=1}^{k} Y_i$ where $Y : \Omega \to \mathbb{N}^{\mathbb{N}}$ is an* i.i.d. *random sequence distributed identically to $\tau_X^{\{1\}}$.*

Proof. When the Bernoulli step size sequence X is *i.i.d.* with $\mathbb{E}X_1 = p \in (0,1)$, we get that $P\left\{\tau_S^{\{1\}} = n\right\} = (1-p)^{n-1}p$ for all $n \in \mathbb{N}$. It follows that

$$P\left\{\tau_S^{\{1\}} < \infty\right\} = P\left(\cup_{n\in\mathbb{N}}\left\{\tau_S^{\{1\}} = n\right\}\right) = \sum_{n\in\mathbb{N}} P\left\{\tau_S^{\{1\}} < \infty\right\} = 1.$$

Hence, the random time $\tau_S^{\{1\}}$ is finite almost surely. We will show that $\tau_S^{\{k\}}$ is finite almost surely for all $k \in \mathbb{N}$ by induction. By induction hypothesis, $\tau_S^{\{k-1\}}$ is finite almost surely. Then $S_{\tau_S^{\{k-1\}}+n} - S_{\tau_S^{\{k-1\}}} = \sum_{j=1}^{n} X_{\tau_S^{\{k-1\}}+j}$ is the sum of n *i.i.d.* Bernoulli random variables, and hence has distribution identical to S_n. Further, This implies that $\tau_S^{\{k\}} = \tau_S^{\{k-1\}} + \tau_S^{\{1\}}$, where $\tau_S^{\{1\}}$ has the identical distribution to $\tau_X^{\{1\}}$ and is independent of $\tau_S^{\{k-1\}}$. □