Lecture-21: Recurrent and transient states

1 Recurrence and Transience

Definition 1.1. For a process $X : \Omega \to \mathcal{X}^{\mathbb{R}_+}$ event A and random variable $Y : \Omega \to \mathbb{R}$, we denote the conditional probability and conditional distribution, given the initial state $\{X_0 = x\}$ by $P_x(A) \triangleq P(A \mid \{X_0 = x\})$ and $\mathbb{E}_x Y = \mathbb{E}[Y \mid \{X_0 = x\}]$ respectively.

Definition 1.2. For a random sequence $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$ with initial state $X_0 = x$,

(i) the **probability of hitting state** y **eventually** is denoted by $f_{xy} \triangleq P_x \left\{ \tau_X^{\{y\},\ell} < \infty \right\}$, and

(ii) the **probability of first visit to state** y **at time** $n \in \mathbb{N}$ is denoted by $f_{xy}^{(n)} \triangleq P_x \left\{ \tau_X^{\{y\},\ell} = n \right\}$.

Remark 1. We can write the finiteness of hitting time $\tau_X^{\{y\},\ell}$ as the disjoint union $\left\{ \tau_X^{\{y\},\ell} < \infty \right\} = \cup_{n \in \mathbb{N}} \left\{ \tau_X^{\{y\},\ell} = n \right\}$. Therefore, $f_{xy} = \sum_{n \in \mathbb{N}} f_{xy}^{(n)}$.

Remark 2. If $f_{xy} = P_x \left\{ \tau_X^{\{y\},\ell} < \infty \right\} = 1$ for all initial states $x \in \mathcal{X}$, then $\tau_X^{\{y\},\ell}$ is almost surely finite and hence a stopping time.

Definition 1.3. From the initial state x, the distribution

(i) for the first hitting time to state y is called the **first passage time distribution** and denoted by $((f_{xy}^{(n)} : n \in \mathbb{N}), 1 - f_{xy})$, and

(ii) for the first return time to state x is called the **first recurrence time distribution** and denoted by $((f_{xx}^{(n)} : n \in \mathbb{N}), 1 - f_{xx})$.

Definition 1.4. A state is called **recurrent** if $f_{xx} = 1$, and is called **transient** if $f_{xx} < 1$.

Definition 1.5. For any state $x \in \mathcal{X}$, the **mean recurrence time** is denoted by $\mu_{xx} \triangleq \mathbb{E}_x \tau_x^{(1)}$.

Remark 3. The mean recurrence time for any transient state is infinite. For any recurrent state $x \in \mathcal{X}$, $\tau_x^{(1)} = \tau_x^{(1)} \mathbb{1}_{\left\{ \tau_X^{(1)} < \infty \right\}} = \sum_{n \in \mathbb{N}} n \mathbb{1}_{\left\{ \tau_x^{(1)} = n \right\}}$ almost surely, and the mean recurrence time is given by $\mu_{xx} = \sum_{n \in \mathbb{N}} n f_{xx}^{(n)}$.

Definition 1.6. For a recurrent state $x \in \mathcal{X}$,

(i) if the mean recurrence time is finite, then the state x is called **positive recurrent**, and

(ii) if the mean recurrence time is infinite, then the state x is called **null recurrent**.

Proposition 1.7. *For a homogeneous discrete Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$, we have*

$$P_x \{ N_y = m \} = \begin{cases} 1 - f_{xy}, & m = 0, \\ f_{xy} f_{yy}^{m-1} (1 - f_{yy}), & m \in \mathbb{N}. \end{cases}$$

Proof. We can write the event of zero visits to state y as $\{ N_y(\infty) = 0 \} = \left\{ \tau_X^{\{y\},\ell} = \infty \right\}$. Further, we can write the event of m visits to state y as

$$\{ N_y(\infty) = m \} = \left\{ \tau_X^{\{y\},m} < \infty \right\} \cap \left\{ \tau_X^{\{y\},m+1} = \infty \right\} = \cap_{j=1}^{m} \left\{ H_X^{\{y\},j} < \infty \right\} \cap \left\{ H_X^{\{y\},m+1} = \infty \right\}, \quad m \in \mathbb{N}.$$

Recall that $(H_X^{\{y\},k} : k \in \mathbb{N})$ is an independent random sequence with $(H_X^{\{y\},k} : k \geqslant 2)$ identically distributed, with $P_x \left\{ H_X^{\{y\},k} = n \right\} = P_y \left\{ \tau_X^{\{y\},\ell} = n \right\}$ for all $k \geqslant 2$. Therefore, we get

$$P_x \{N_y = m\} = P_x \left\{ H_X^{\{y\},1} < \infty \right\} \prod_{j=2}^{m} P_x \left\{ H_X^{\{y\},j} < \infty \right\} P_x \left\{ H_X^{\{y\},m+1} = \infty \right\} = f_{xy} f_{yy}^{m-1} (1 - f_{yy}).$$

□

Corollary 1.8. *For a homogeneous Markov chain X, we have* $P_x \{N_y < \infty\} = \mathbb{1}_{\{f_{yy} < 1\}} + (1 - f_{xy}) \mathbb{1}_{\{f_{yy} = 1\}}$.

Proof. We can write the event $\{N_y < \infty\}$ as disjoint union of events $\{N_y = n\}$, to get the result. □

Remark 4. For a time homogeneous Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$, we have
(i) $P_x \{N_y = \infty\} = f_{xy} \mathbb{1}_{\{f_{yy} = 1\}}$, and
(ii) $P_y \{N_y = \infty\} = \mathbb{1}_{\{f_{yy} = 1\}}$.

Corollary 1.9. *The mean number of visits to state y, starting from a state x is* $\mathbb{E}_x N_y = \frac{f_{xy}}{1 - f_{yy}} \mathbb{1}_{\{f_{yy} < 1\}} + \infty \mathbb{1}_{\{f_{xy} > 0, f_{yy} = 1\}}$.

Remark 5. For any $x \in \mathcal{X}$, we have $\mathbb{E}_x N_x = \frac{f_{xx}}{1 - f_{xx}} \mathbb{1}_{\{f_{xx} < 1\}} + \infty \mathbb{1}_{\{f_{xx} = 1\}}$. That is, the mean number of visits to initial state x is finite iff the state x is transient.

Remark 6. In particular, this corollary implies the following consequences.

i. A transient state is visited a finite amount of times almost surely. This follows from Corollary 1.8, since $P_x \{N_y < \infty\} = 1$ for all transient states $y \in \mathcal{X}$ and any initial state $x \in \mathcal{X}$.

ii. A recurrent state is visited infinitely often almost surely. This also follows from Corollary 1.8, since $P_y \{N_y < \infty\} = 0$ for all recurrent states $y \in \mathcal{X}$.

iii. In a finite state Markov chain, not all states may be transient.

Proof. To see this, we assume that for a finite state space $\mathcal{X}$, all states $y \in \mathcal{X}$ are transient. Then, we know that N_y is finite almost surely for all states $y \in \mathcal{X}$. It follows that, for any initial state $x \in \mathcal{X}$

$$0 \leqslant P_x \left\{ \sum_{y \in \mathcal{X}} N_y = \infty \right\} = P_x (\cup_{y \in \mathcal{X}} \{N_y = \infty\}) \leqslant \sum_{y \in \mathcal{X}} P_x \{N_y = \infty\} = 0.$$

It follows that $\sum_{x \in \mathcal{X}} N_x$ is also finite almost surely for all states $y \in \mathcal{X}$ for finite state space $\mathcal{X}$. However, we know that $\sum_{x \in \mathcal{X}} N_x = \sum_{k \in \mathbb{N}} \sum_{x \in \mathcal{X}} 1_{\{X_k = x\}} = \infty$. This leads to a contradiction. □

Proposition 1.10. *For a homogeneous DTMC* $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$*, a state x is recurrent iff* $\sum_{k \in \mathbb{N}} p_{xx}^{(k)} = \infty$*, and transient iff* $\sum_{k \in \mathbb{N}} p_{xx}^{(k)} < \infty$.

Proof. Recall that if the mean recurrence time to a state x is $\mathbb{E}_x N_x = \sum_{k \in \mathbb{N}} p_{xx}^k$ finite then the state is transient and infinite if the state is recurrent. □

Corollary 1.11. *For a transient state* $y \in \mathcal{X}$*, the following limits hold* $\lim_{n \to \infty} p_{xy}^{(n)} = 0$*, and* $\lim_{n \to \infty} \frac{\sum_{k=1}^{n} p_{xy}^{(k)}}{n} = 0$.

Proof. For a transient state $y \in \mathcal{X}$ and any state $x \in \mathcal{X}$, we have $\mathbb{E}_x N_y = \sum_{n \in \mathbb{N}} p_{xy}^{(n)} < \infty$. Since the series sum is finite, it implies that the limiting terms in the sequence $\lim_{n \to \infty} p_{xy}^{(n)} = 0$. Further, we can write $\sum_{k=1}^{n} p_{xy}^{(k)} \leqslant \mathbb{E}_x N_y \leqslant M$ for some $M \in \mathbb{N}$ and hence $\lim_{n \to \infty} \frac{\sum_{k=1}^{n} p_{xy}^{(k)}}{n} = 0$. □

Lemma 1.12. *For any state $y \in \mathcal{X}$, let $(H_X^{\{y\},\ell} : \ell \in \mathbb{N})$ be the sequence of almost surely finite inter-visit times to state y, and $N_y(n) = \sum_{k=1}^n 1_{\{X_k = y\}}$ be the number of visits to state y in n times. Then, $N_y(n)+1$ is a finite mean stopping time with respect to the sequence $(H_X^{\{y\},\ell} : \ell \in \mathbb{N})$.*

Proof. We first observe that $N_y(n)+1 \leqslant n+1$ and hence has a finite mean for each $n \in \mathbb{N}$. Further, we observe that $\{N_y(n)+1 = k\}$ can be completely determined by observing $H_X^{\{y\},1}, \dots, H_X^{\{y\},k}$. To see this, we notice that

$$\{N_y(n)+1 = k\} = \left\{ \sum_{\ell=1}^{k-1} H_X^{\{y\},\ell} \leqslant n < \sum_{\ell=1}^{k} H_X^{\{y\},\ell} \right\} \in \sigma(H_X^{\{y\},1}, \dots, H_X^{\{y\},k}).$$

□

Theorem 1.13. *Let $x, y \in \mathcal{X}$ be such that $f_{xy} = 1$ and y is recurrent. Then, $\lim_{n\to\infty} \frac{\sum_{k=1}^n p_{xy}^{(k)}}{n} = \frac{1}{\mu_{yy}}$.*

Proof. Let $y \in \mathcal{X}$ be recurrent. The proof consists of three parts. In the first two parts, we will show that starting from the state y, we have the limiting empirical average of mean number of visits to state y is $\lim_{n\to\infty} \frac{1}{n}\mathbb{E}_y N_y(n) = \frac{1}{\mu_{yy}}$. In the third part, we will show that for any starting state $x \in \mathcal{X}$ such that $f_{xy} = 1$, we have the limiting empirical average of mean number of visits to state y is $\lim_{n\to\infty} \frac{1}{n}\mathbb{E}_x N_y(n) = \frac{1}{\mu_{yy}}$.

Lower bound: We observe that $N_y(n)+1$ is a stopping time with respect to inter-visit times $(H_X^{\{y\},\ell} : \ell \in \mathbb{N})$ from Lemma 1.12. Further, we have $\sum_{\ell=1}^{N_y(n)+1} H_X^{\{y\},\ell} > n$. Applying Wald's Lemma to the random sum $\sum_{\ell=1}^{N_y(n)+1} H_X^{\{y\},\ell}$, we get $\mathbb{E}_y(N_y(n)+1)\mu_{yy} > n$. Taking limits, we obtain $\liminf_{n\in\mathbb{N}} \frac{\sum_{k=1}^n p_{yy}^{(k)}}{n} \geqslant \frac{1}{\mu_{yy}}$.

Upper bound: Given a fixed positive integer $M \in \mathbb{N}$, we define truncated recurrence times

$$\bar{H}_X^{\{y\},\ell} \triangleq M \wedge H_X^{\{y\},\ell} \text{ for all } \ell \in \mathbb{N}.$$

Since $H_X^{\{y\}}$ is *i.i.d.* given the initial state y, then so is $\bar{H}_X^{\{y\},1}$ and $\bar{H}_X^{\{y\},\ell} \leqslant H_X^{\{y\},\ell}$ for all $\ell \in \mathbb{N}$. We define the mean of the truncated recurrence times as $\bar{\mu}_{yy} \triangleq \mathbb{E}_y \bar{H}_X^{\{y\},1}$. From the monotonicity of truncation, we get $\bar{\mu}_{yy} \leqslant \mu_{yy}$.

We define the random variable $\bar{\tau}_X^{\{y\},k} \triangleq \sum_{\ell=1}^k \bar{H}_X^{\{y\},\ell}$ for all $k \in \mathbb{N}$, and $\bar{\tau}_X^{\{y\},k} \leqslant \tau_X^{\{y\},k}$ for all $k \in \mathbb{N}$. We can define the associated counting process that counts number of truncated recurrences in first n steps as $\bar{N}_y(n) \triangleq \sum_{k\in\mathbb{N}} \mathbb{1}_{\left\{\bar{\tau}_X^{\{y\},k} \leqslant n\right\}}$ for all $n \in \mathbb{N}$. Further, we have

$$\sum_{\ell=1}^{\bar{N}_y(n)+1} \bar{H}_X^{\{y\},\ell} = \bar{\tau}_X^{\{y\},\bar{N}_y(n)+1} = \bar{\tau}_X^{\{y\},\bar{N}_y(n)} + \bar{H}_X^{\{y\},\bar{N}_y(n)+1} \leqslant n + M.$$

Since $\bar{N}_y(n)+1$ is a stopping time with respect to *i.i.d.* process $\bar{H}_X^{\{y\}}$, and $\bar{N}_y(n) \geqslant N_y(n)$ sample path wise. From Wald's Lemma, we get

$$\mathbb{E}_y(N_y(n)+1)\bar{\mu}_{yy} \leqslant \mathbb{E}_y(\bar{N}_y(n)+1)\bar{\mu}_{yy} \leqslant n + M.$$

Taking limits, we obtain $\limsup_{n\in\mathbb{N}} \frac{\sum_{k=1}^n p_{xy}^{(k)}}{n} \leqslant \frac{1}{\bar{\mu}_{yy}}$. Letting M grow arbitrarily large, we obtain the upper bound.

Starting from x: Further, we observe that $p_{xy}^{(k)} = \sum_{s=0}^{k-1} f_{xy}^{(k-s)} p_{yy}^{(s)}$. Since $1 = f_{xy} = \sum_{k\in\mathbb{N}} f_{xy}^{(k)}$, we have

$$\sum_{k=1}^n p_{xy}^{(k)} = \sum_{k=1}^n \sum_{s=0}^{k-1} f_{xy}^{(k-s)} p_{yy}^{(s)} = \sum_{s=0}^{n-1} p_{yy}^{(s)} \sum_{k-s=1}^{n-s} f_{xy}^{(k-s)} = \sum_{s=0}^{n-1} p_{yy}^{(s)} - \sum_{s=0}^{n-1} p_{yy}^{(s)} \sum_{k>n-s} f_{xy}^{(k)}.$$

Since the series $\sum_{k\in\mathbb{N}} f_{xy}^{(k)}$ converges, we get $\lim_{n\to\infty} \frac{\sum_{k=1}^n p_{xy}^{(k)}}{n} = \lim_{n\to\infty} \frac{\sum_{k=1}^n p_{yy}^{(k)}}{n}$.

□