Lecture-24: Poisson Processes

1 Simple point processes

Consider the d-dimensional Euclidean space $\mathbb{R}^d$. The collection of Borel measurable subsets $\mathcal{B}(\mathbb{R}^d)$ of the above Euclidean space is generated by sets $B(x) \triangleq \left\{y \in \mathbb{R}^d : y_i \leqslant x_i\right\}$ for $x \in \mathbb{R}^d$.

Definition 1.1. A **simple point process** is a random countable collection of *distinct points* $S : \Omega \to \mathcal{X}^{\mathbb{N}}$, such that the distance $\|S_n\| \to \infty$ as $n \to \infty$.

Remark 1. For any simple point process S, we have $P(\{S_n = S_m \text{ for any } n \neq m\}) = 0$.

Example 1.2 (Simple point process on the half-line). We can simplify this definition for $d = 1$. When $\mathcal{X} = \mathbb{R}_+$, one can order the points of the process $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ to get ordered process $\tilde{S} : \Omega \to \mathbb{R}_+^{\mathbb{N}}$, such that $\tilde{S}_n = S_{(n)}$ is the nth order statistics of S. That is, $S_{(0)} \triangleq 0$, and $S_{(n)} \triangleq \inf\left\{S_k > S_{(n-1)} : k \in \mathbb{N}\right\}$. such that $S_{(1)} < S_{(2)} < \cdots < S_{(n)} < \ldots$, and $\lim_{n\in\mathbb{N}} S_{(n)} = \infty$. We will call this an arrival process. The Borel measurable sets for $\mathbb{R}_+$ are generated by the collection of half-open intervals $\{(0,t] : t \in \mathbb{R}_+\}$.

Point processes can model many interesting physical processes.

1. Arrivals at classrooms, banks, hospital, supermarket, traffic intersections, airports etc.
2. Location of nodes in a network, such as cellular networks, sensor networks, etc.

Definition 1.3. Corresponding to a point process $S : \Omega \to \mathcal{X}^{\mathbb{N}}$, we denote the number of points in a set $A \in \mathcal{B}(\mathcal{X})$ by

$$N(A) = \sum_{n\in\mathbb{N}} \mathbb{1}_{\{S_n \in A\}}, \text{ where we have } N(\emptyset) = 0.$$

The resulting process $N : \Omega \to {\mathbb{Z}_+}^{\mathcal{B}(\mathcal{X})}$ is called a **counting process** for the point process $S : \Omega \to \mathcal{X}^{\mathbb{N}}$.

Definition 1.4. A counting process is **simple** if the underlying point process is simple.

Remark 2. For a simple counting process N, we have $N(\{x\}) \leqslant 1$ almost surely for all $x \in \mathcal{X}$.

Remark 3. Let $N : \Omega \to {\mathbb{Z}_+}^{\mathcal{B}(\mathcal{X})}$ be the counting process for the point process $S : \Omega \to \mathcal{X}^{\mathbb{N}}$.

i. Note that the point process S and the counting process N carry the same information.

ii. The distribution of point process S is completely characterized by the finite dimensional distributions $(N(A_1), \ldots, N(A_k) :$ bounded $A_1, \ldots, A_k \in \mathcal{B}(\mathcal{X}))$ for some finite $k \in \mathbb{N}$.

Example 1.5 (Simple point process on the half-line). The number of points in the half-open interval $(0,t]$ is denoted by

$$N(t) \triangleq N((0,t]) = \sum_{n\in\mathbb{N}} \mathbb{1}_{\{S_n\in(0,t]\}}.$$

Since the Borel measurable sets $\mathcal{B}(\mathbb{R}_+)$ are generated by half-open intervals $\{(0,t] : t \in \mathbb{R}_+\}$, we denote the counting process by $N : \Omega \to \mathbb{Z}_+{}^{\mathbb{R}_+}$, where $N(t) = N((0,t])$. For $s < t$, the number of points in interval $(s,t]$ is $N((s,t]) = N((0,t]) - N((0,s]) = N(t) - N(s)$.

Theorem 1.6 (Rényi). *Distribution of a simple point process $S : \Omega \to \mathcal{X}^{\mathbb{N}}$ on a locally compact second countable space $\mathcal{X}$ is completely determined by void probabilities $(P\{N(A) = 0\} : A \in \mathcal{B}(\mathcal{X}))$.*

Proof. It suffices to show that the finite dimensional distributions of S on locally compact sets are characterized by void probabilities.

Step 1: We will show this by induction on the number of points k in a bounded set $A \in \mathcal{B}$. Let $A_1,\dots,A_k,B \in \mathcal{B}(\mathcal{X})$ locally compact, then we will show that $u_k \triangleq P(\cap_{i=1}^k \{N(A_i) > 0\} \cap \{N(B) = 0\})$ can be computed from void probabilities. From $k = 1$, we have

$$P\{N(A_1) > 0, N(B) = 0\} = P\{N(B) = 0\} - P\{N(B \cup A_1) = 0\}.$$

The induction can be proved by the recursive relation

$$u_k = P(\cap_{i=1}^{k-1} \{N(A_i) > 0\} \cap \{N(B) = 0\}) - P(\cap_{i=1}^{k} \{N(A_i) > 0\} \cap \{N(A_k \cup B) = 0\}).$$

Step 2: For any locally compact set $B \in \mathcal{B}(\mathcal{X})$, there exists a sequence of nested partitions $B_n \triangleq (B_{n,j} : j \in [k_n])$ that eventually separates the points in $S \cap B$ as $n \to \infty$. We define the number of subsets of partition $(B_{n,j} : j \in [k_n])$ that consist of at least point in $S \cap B$, as

$$H_n(B) \triangleq \sum_{j=1}^{k_n} \mathbb{1}_{\{N(B_{n,j})>0\}},$$

where $H_n(B) \uparrow N(B)$ almost surely.

Step 3: We now show that for all locally compact sets $B_1,\dots,B_k \in \mathcal{B}(\mathcal{X})$ and $j_1,\dots,j_k \in \mathbb{N}$, the probability $P(\cap_{i=1}^k \{H_n(B_i) = j_i\})$ can be expressed in terms of void probabilities. We observe that

$$P\{H_n(B) = j\} = \sum_{T\subseteq[k_n]:|T|=j} P(\cap_{j\in T} \{N(B_{n,j}) > 0\} \cap \left\{N(\cup_{j\notin T} B_{n,j}) = 0\right\}).$$

This can be expressed in terms of void probabilities by Step 1.

Step 4: For a simple point process, we have the following almost sure limit

$$\lim_n \cap_{i=1}^k \{H_n(B_i) = j_i\} = \cap_{i=1}^k \{N(B_i) = j_i\}.$$

The result follows from the continuity of probability.

□

Definition 1.7. A non-negative integer valued random variable $N : \Omega \to \mathbb{Z}_+$ is called **Poisson** if for some constant $\lambda > 0$, we have

$$P\{N = n\} = e^{-\lambda}\frac{\lambda^n}{n!}.$$

Remark 4. It is easy to check that $\mathbb{E}N = \mathrm{Var}[N] = \lambda$. Furthermore, the moment generating function $M_N(t) = \mathbb{E}e^{tN} = e^{\lambda(e^t-1)}$ exists for all $t \in \mathbb{R}$.

Corollary 1.8. *A simple counting process $N : \Omega \to \mathbb{Z}_+^{\mathcal{B}(\mathcal{X})}$ has Poisson marginal distribution if and only if void probabilities are exponential.*

Proof. From Rényi's theorem, finite dimensional distribution of counting process of simple point processes is determined by void probabilities. It is clear that if marginal distribution is Poisson, then the void probability $P\{N(A) = 0\} = e^{-\Lambda(A)}$ is exponential.

From the fact that void probabilities are exponentially distributed, we will show that for any finite, bounded, and disjoint sets $B_1, \dots, B_k \in \mathcal{B}(\mathcal{X}))$, we have

$$P(\cap_{i=1}^k \{N(B_i) = 0\}) = e^{-\Lambda(\cup_{i=1}^k B_i)} = \prod_{i=1}^k e^{-\Lambda(B_i)} = \prod_{i=1}^k P\{N(B_i) = 0\}.$$

We further observe that $H_n(B) = \sum_{j=1}^{k_n} \mathbb{1}_{\{N(B_{n,j})>0\}}$ is the sum of k_n independent Bernoulli random variables with success probability $1 - e^{-\Lambda(B_{n,j})}$. Therefore,

$$P\{H_n(B) = m\} = \sum_{T \in [k_n]:|T|=m} \prod_{j \in T} (1 - e^{-\Lambda(B_{n,j})}) \prod_{j \notin T} e^{-\Lambda(B_{n,j})}.$$

Recall that $H_n(B) \uparrow N(B)$ as $n \to \infty$ in the proof of Rényi's Theorem, and $\lim_{n \in \mathbb{N}} |B_{n,j}| = 0$. Taking limit $n \to \infty$ on both sides of the above equation, we get it follows that

$$P\{N(B) = m\} = e^{-\Lambda(B)} \sum_{T \in [k_n]:|T|=m} \prod_{j \in T} \Lambda(B_{n,j}) = e^{-\Lambda(B)} \frac{\Lambda(B)^m}{m!}.$$

□

Definition 1.9. A counting process $N : \Omega \to \mathbb{Z}_+^{\mathcal{B}(\mathcal{X})}$ has the **completely independence property**, if for any collection of finite disjoint and bounded sets $A_1, \dots, A_k \in \mathcal{B}(\mathcal{X})$, the vector $(N(A_1), \dots, N(A_k)) : \Omega \to \mathbb{Z}_+^k$ is independent. That is,

$$P\left(\bigcap_{i=1}^k \{N(A_i) = n_i\}\right) = \prod_{i=1}^k P\{N(A_i) = n_i\}, \quad n \in \mathbb{Z}_+^k.$$

2 Poisson point process

Remark 5. Recall that $|A| = \int_{x \in A} dx$ is the volume of the set $A \in \mathcal{B}(\mathbb{R}^d)$ and for any such A, the intensity measure of this set is scaled volume

$$\Lambda(A) = \int_{x \in A} \lambda(x) dx,$$

for the intensity density $\lambda : \mathbb{R}^d \to \mathbb{R}_+$. If the intensity density $\lambda(x) = \lambda$ for all $x \in \mathbb{R}^d$, then $\Lambda(A) = \lambda |A|$. In particular for partition $A_1, \dots, A_k$ for a set A, we have $\Lambda(A) = \sum_{i=1}^k \Lambda(A_i)$.

Definition 2.1. A simple point process $S : \Omega \to \mathcal{X}^{\mathbb{N}}$ is **Poisson point process**, if the associated counting process $N : \Omega \to \mathbb{Z}_+^{\mathcal{B}(\mathcal{X})}$ has complete independence property and the marginal distributions are Poisson.

Definition 2.2. The **intensity measure** $\Lambda : \mathcal{B}(\mathcal{X}) \to \mathbb{R}_+$ of Poisson process S is defined by $\Lambda(A) \triangleq \mathbb{E}N(A)$ for all bounded $A \in \mathcal{B}(\mathcal{X})$.

Remark 6. That is, for a Poisson process with intensity measure Λ, $k \in \mathbb{Z}_+$, and bounded mutually disjoint sets $A_1, \dots, A_k \in \mathcal{B}(\mathcal{X})$, we have

$$P\left(\cap_{i=1}^k \{N(A_i) = n_i\}\right) = \prod_{i=1}^k \left(e^{-\Lambda(A_i)} \frac{\Lambda(A_i)^{n_i}}{n_i!}\right), \quad n \in \mathbb{Z}_+^k.$$

Definition 2.3. If the intensity measure Λ of a Poisson process S satisfies $\Lambda(A) = \lambda |A|$ for all bounded $A \in \mathcal{B}(\mathcal{X})$, then we call S a **homogeneous Poisson point process** and λ is its intensity.

3 Equivalent characterizations

Theorem 3.1 (Equivalences). *Following are equivalent for a simple counting process $N : \Omega \to \mathbb{Z}_+{}^{\mathcal{B}(\mathcal{X})}$.*

i_ Process N is Poisson with locally finite intensity measure Λ.

ii_ For each bounded $A \in \mathcal{B}(\mathcal{X})$, we have $P\{N(A) = 0\} = e^{-\Lambda(A)}$.

iii_ For each bounded $A \in \mathcal{B}(\mathcal{X})$, the number of points $N(A)$ is a Poisson with parameter $\Lambda(A)$.

iv_ Process N has the completely independence property, and $\mathbb{E}N(A) = \Lambda(A)$.

Proof. We will show that $i\_ \implies ii\_ \implies iii\_ \implies iv\_ \implies i\_$.

$i \implies ii\_$ It follows from the definition of Poisson point processes and definition of Poisson random variables.

$ii \implies iii\_$ From Corollary 1.8, we know that if void probabilities are exponential, then the marginal distributions are Poisson.

$iii \implies iv\_$ We will show this in two steps.

Mean: Since the distribution of random variable $N(A)$ is Poisson, it has mean $\mathbb{E}N(A) = \Lambda(A)$.

CIP: For disjoint and bounded $A_1, \dots, A_k \in \mathcal{B}$ and $A = \cup_{i=1}^k A_i$, we have $N(A) = N(A_1) + \dots N(A_1)$. Taking expectations on both sides, and from the linearity of expectation, we get

$$\Lambda(A) = \Lambda(A_1) + \cdots + \Lambda(A_k).$$

From the number of partitions $n_1 + \cdots + n_k = n$, we can write

$$P\{N(A) = n\} = \sum_{n_1+\cdots+n_k=n} P\{N(A_1) = n_1, \dots, N(A_k) = n_k\}.$$

Using the definition of Poisson distribution, we can write the LHS of the above equation as

$$P\{N(A) = n\} = e^{-\Lambda(A)} \frac{\Lambda(A)^n}{n!} = \prod_{i=1}^k e^{-\Lambda(A_i)} \frac{(\sum_{i=1}^k \Lambda(A_i))^n}{n!}.$$

Since the expansion of $(a_1 + \cdots + a_k)^n = \sum_{n_1+\cdots+n_k=n} \binom{n}{n_1,\dots,n_k} \prod_{i=1}^k a_i^{n_i}$, we get

$$P\{N(A) = n\} = \frac{1}{n!} \sum_{n_1+\cdots+n_k=n} \binom{n}{n_1,\dots,n_k} \prod_{i=1}^k e^{-\Lambda(A_i)} \Lambda(A_i))^{n_i} = \sum_{n_1+\cdots+n_k=n} \left(\prod_{i=1}^k e^{-\Lambda(A_i)} \frac{\Lambda(A_i))^{n_i}}{n_i!} \right).$$

Equating each term in the summation, we get

$$P\{N(A_1) = n_1, \dots, N(A_k) = n_k\} = \prod_{i=1}^k P\{N(A_i) = n_i\}.$$

$iv \implies i\_$ Since void probabilities describe the entire distribution, it suffices to show that $P\{N(A) = 0\} = e^{-\Lambda(A)}$ for all bounded $A \in \mathcal{B}$.

□

Corollary 3.2 (Poisson process on the half-line). *A random process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ indexed by time $t \in \mathbb{Z}_+$ is the counting process associated with a one-dimensional Poisson process $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ having intensity measure Λ iff*

(a) Starting with $N(0) = 0$, the process $N(t)$ takes a non-negative integer value for all $t \in \mathbb{R}_+$;

(b) the increment $N(t+s) - N(t)$ is surely nonnegative for any $s \in \mathbb{R}_+$;

(c) *the increments* $N(t_1), N(t_2) - N(t_1), \ldots, N(t_n) - N(t_{n-1})$ *are independent for any* $0 < t_1 < t_2 < \cdots < t_{n-1} < t_n$*;*

(d) *the increment* $N(t+s) - N(t)$ *is distributed as Poisson random variable with parameter* $\Lambda((t, t+s])$*.*

The Poisson process is homogeneous with intensity λ*, iff in addition to conditions* $(a),(b),(c)$*, the distribution of the increment* $N(t+s) - N(t)$ *depends on the value* $s \in \mathbb{R}_+$ *but is independent of* $t \in \mathbb{R}_+$*. That, is the increments are stationary.*

Proof. We have already seen that definition of Poisson processes implies all four conditions. Conditions (a) and (b) imply that N is a simple counting process on the half-line, condition (c) is the complete independence property of the point process, and condition (d) provides the intensity measure. The result follows from the equivalence *iv*₋ in Theorem 3.1. $\square$