Lecture-25: Poisson processes: Conditional distribution

1 Joint conditional distribution of points in a finite window

Let $\mathcal{X} = \mathbb{R}^d$ be a d-dimensional Euclidean space, and $S : \Omega \to \mathcal{X}^{\mathbb{N}}$ be a Poisson point process with intensity measure $\Lambda : \mathcal{B}(\mathcal{X}) \to \mathbb{R}_+$ and associated counting process $N : \Omega \to \mathbb{Z}_+^{\mathcal{B}(\mathcal{X})}$.

Remark 1. Since S is a simple point process, each point S_n is unique. Therefore, we can identify S as a random set of points in $\mathcal{X}$ and $S \cap A$ is the random set of points in A. It follows that $\{N(A) = n\} = \{|S \cap A| = n\}$.

Remark 2. Let $(A_1, \dots, A_k) \in \mathcal{B}(\mathcal{X})^k$ be disjoint bounded subsets such that $A = \cup_{i=1}^k A_i \in \mathcal{B}(\mathcal{X})$. From the disjointness of A_i, we have

$$N(A) = \sum_{n \in \mathbb{N}} \mathbb{1}_{\cup_{i=1}^k A_i}(S_n) = \sum_{n \in \mathbb{N}} \sum_{i=1}^k \mathbb{1}_{A_i}(S_n) = \sum_{i=1}^k \sum_{n \in \mathbb{N}} \mathbb{1}_{A_i}(S_n) = \sum_{i=1}^k N(A_i).$$

The result follows from the linearity of expectations, such that $\Lambda(A) = \mathbb{E}N(A) = \sum_{i=1}^k \mathbb{E}N(A_i) = \sum_{i=1}^k \Lambda(A_i)$.

Proposition 1.1. *Let $k \in \mathbb{N}$ be any positive integer. For a Poisson point process $S : \Omega \to \mathcal{X}^{\mathbb{N}}$ with intensity measure $\Lambda : \mathcal{B}(\mathcal{X}) \to \mathbb{R}_+$, consider a bounded subset $A \in \mathcal{B}(\mathcal{X})$ and subsets $(A_1, \dots, A_k) \in \mathcal{B}(\mathcal{X})^k$ that partition A. Let $n_1, \dots, n_k \in \mathbb{Z}_+$ such that $n_1 + \cdots + n_k = n$, then*

$$P(\{N(A_1) = n_1, \dots, N(A_k) = n_k\} \mid \{N(A) = n\}) = \binom{n}{n_1, \dots, n_k} \prod_{i=1}^k \left(\frac{\Lambda(A_i)}{\Lambda(A)}\right)^{n_i}. \tag{1}$$

Proof. From the definition of conditional probability, we can write the conditional probability on LHS as the ratio

$$\frac{P(\{N(A_1) = n_1, \dots, N(A_k) = n_k\} \cap \{N(A) = n\})}{P(\{N(A) = n\})}.$$

Since $\cap_{i=1}^k \{N(A_i) = n_i\} \subseteq \{N(A) = n\}$, we can rewrite the RHS of the above equation as

$$\frac{P(\{N(A_1) = n_1, \dots, N(A_k) = n_k\})}{P(\{N(A) = n\})}.$$

From the definition of joint distribution of $(N(A_1), \dots, N(A_k))$ for disjoint bounded $A_1, \dots, A_k \in \mathcal{B}(\mathcal{X})$, we can rewrite the RHS of the above equation as

$$\frac{P(\{N(A_1) = n_1, \dots, N(A_k) = n_k\})}{P(\{N(A) = n\})} = \frac{\prod_{i=1}^k e^{-\Lambda(A_i)} \frac{\Lambda(A_i)^{n_i}}{n_i!}}{e^{-\Lambda(A)} \frac{\Lambda(A)^n}{n!}} = \binom{n}{n_1, \dots, n_k} \prod_{i=1}^k \left(\frac{\Lambda(A_i)}{\Lambda(A)}\right)^{n_i}.$$

The result follows from the fact that the intensity measure add over disjoint sets, i.e. $\Lambda(A) = \sum_{i=1}^k \Lambda(A_i)$. $\square$

Remark 3. Consider a Poisson point process $S : \Omega \to \mathcal{X}^{\mathbb{N}}$ with intensity measure $\Lambda : \mathcal{B}(\mathcal{X}) \to \mathbb{R}_+$ and counting process $N : \Omega \to \mathbb{Z}_+^{\mathcal{B}(\mathcal{X})}$. Let $(A_1, \dots, A_k) \in \mathcal{B}(\mathcal{X})^k$ be disjoint bounded subsets such that $A = \cup_{i=1}^k A_i \in \mathcal{B}(\mathcal{X})$.

i_ Defining $p_i \triangleq \frac{\Lambda(A_i)}{\Lambda(A)}$, we see that $(p_1, \dots, p_k)$ is a probability distribution. We also observe that

$$p_i = P(\{N(A_i) = 1\} \mid \{N(A) = 1\}) = P(\{|S \cap A_i| = 1\} \mid \{|S \cap A| = 1\}).$$

When $N(A) = 1$, we can call the point of S in A as S_1 without any loss of generality. That is, if we call $\{S_1\} = S \cap A$, then we have

$$p_i = P(\{S_1 \in A_i\} \mid \{S_1 \in A\}).$$

Similarly, when $N(A) = n_i$, we call the points of S in A as $S_1, \dots, S_{n_i}$ and denote $S \cap A = \{S_1, \dots, S_{n_i}\}$. For this case, we observe

$$\begin{aligned} P(\{N(A_i) = n_i\} \mid \{N(A) = n_i\}) &= P(\{|S \cap A_i| = n_i\} \mid \{|S \cap A| = n_i\}) = p_i^{n_i} \\ &= P(\cap_{j=1}^{n_i} \{S_j \in A_i\} \mid \{\{S_1, \dots, S_{n_i}\} = S \cap A\}) = \prod_{j=1}^{n_i} P(\{S_j \in A_i\} \mid \{S_j \in A\}). \end{aligned}$$

ii_ We can rewrite the Equation (1) as a multinomial distribution, where

$$P(\{N(A_1) = n_1, \dots, N(A_k) = n_k\} \mid \{N(A) = n\}) = \binom{n}{n_1, \dots, n_k} p_1^{n_1} \dots p_k^{n_k}.$$

iii_ We define $\mathcal{P}_k(E, n_1, \dots, n_k)$ to be the collection of all k-partitions $(E_1, \dots, E_k)$ of any finite set $E \subseteq \mathbb{N}$ such that $|E_i| = n_i$ for $i \in [k]$. Then, the multinomial coefficient accounts for number of partitions of n points into sets with $n_1, \dots, n_k$ points. That is,

$$\binom{n}{n_1, \dots, n_k} = |\mathcal{P}_k([n], n_1, \dots, n_k)|.$$

iv_ Recall that the event $\{N(A_i) = n_i\} = \{|S \cap A_i| = n_i\}$. Hence, we can write

$$\begin{aligned} P(\cap_{i=1}^k \{|S \cap A_i| = n_i\} \mid \{|S \cap A| = n\}) &= \binom{n}{n_1, \dots, n_k} p_1^{n_1} \dots p_k^{n_k} \\ &= \sum_{(E_1, \dots, E_k) \in \mathcal{P}_k(S \cap A, n_1, \dots, n_k)} \prod_{i=1}^k \prod_{S_j \in E_i} P(\{S_j \in A_i\} \mid \{S_j \in A\}). \end{aligned}$$

v_ When $N(A) = n$, we denote $S \cap A$ by $E = \{S_1, \dots, S_n\}$ without any loss of generality. We further observe that when $N(A_i) = n_i$ for all $i \in [k]$, then $(S \cap A_1, \dots, S \cap A_k) \in \mathcal{P}_k(S \cap A, n_1, \dots, n_k)$. Therefore, we can re-write the event

$$\cap_{i=1}^k \{N(A_i) = n_i\} = \cap_{i=1}^k \{|S \cap A_i| = n_i\} = \cup_{(E_1, \dots, E_k) \in \mathcal{P}_k(S \cap A, n_1, \dots, n_k)} (\cap_{i=1}^k \{S \cap A_i = E_i\}).$$

That is, we can write the conditional probability

$$\begin{aligned} P(\cap_{i=1}^k \{N(A_i) = n_i\} \mid \{N(A) = n\}) &= \sum_{(E_1, \dots, E_k) \in \mathcal{P}_k(S \cap A, n_1, \dots, n_k)} P(\cap_{i=1}^k \{S \cap A_i = E_i\} \mid \{S \cap A = E\}) \\ &= \sum_{(E_1, \dots, E_k) \in \mathcal{P}_k(S \cap A, n_1, \dots, n_k)} P(\cap_{i=1}^k \cap_{S_j \in E_i} \{S_j \in A_i\} \mid \{S \cap A = E\}). \end{aligned}$$

vi. Equating the RHS of the above equation term-wise, we obtain that conditioned on each of these points falling inside the window A, the conditional probability of each point falling in partition A_i is independent of all other points and given by p_i. That is, we have

$$P(\cap_{i=1}^k \cap_{S_j \in E_i} \{S_j \in A_i\} \mid \{S \cap A = E\}) = \prod_{i=1}^k \prod_{S_j \in E_i} P(\{S_j \in A_i\} \mid \{S_j \in A\}) = \prod_{i=1}^k p_i^{n_i} = \prod_{i=1}^k \left(\frac{\Lambda(A_i)}{\Lambda(A)}\right)^{n_i}.$$

It means that given n points in the window A, the location of these points are independently and identically distributed in A according to the distribution $\frac{\Lambda(\cdot)}{\Lambda(A)}$.

vii. If the Poisson process is homogeneous, the distribution is uniform over the window A.

viii. For a Poisson process with intensity measure Λ and any bounded set $A \in \mathcal{B}$, we have $N(A)$ a Poisson random variable with parameter $\Lambda(A)$. Given $N(A)$, the location of all the points in $S \cap A$ are *i.i.d.* with density $\frac{\lambda(x)}{\Lambda(A)}$ for all $x \in A$.

Remark 4 (Simulating a homogeneous Poisson point process). If we are interested in simulating a two dimensional homogeneous Poisson point process with density λ in a uniform square $A = [0,1] \times [0,1]$. Then, we first generate the random variable $N(A) : \Omega \to \mathbb{Z}_+$ that takes value n with probability $e^{-\lambda}\frac{\lambda^n}{n!}$. Next, for each of the $N(A) = n$ points, we generate the location $(X_i, Y_i) \in \mathbb{R}^2$ uniformly at random. That is, $X : \Omega \to [0,1]^n$ and $Y : \Omega \to [0,1]^n$ are independent *i.i.d.* uniform sequences.

Corollary 1.2. *For a homogeneous Poisson point process on the half-line with ordered set of points* $(S_{(n)} \in \mathbb{R}_+ : n \in \mathbb{N})$*, we can write the conditional density of ordered points* $(S_{(1)}, \dots, S_{(k)})$ *given the event* $\{N(t) = k\}$ *as ordered statistics of* i.i.d. *uniformly distributed random variables. Specifically, we have*

$$f_{S_{(1)},\dots,S_{(k)} \mid N(t)=k}(t_1, \dots, t_k) = k! \frac{\mathbb{1}_{\{0 < t_1 \leqslant \dots \leqslant t_k \leqslant t\}}}{t^k}.$$

Proof. Given $\{N(t) = k\}$, we can denote the points of the Poisson process in $(0,t]$ by $S_1, \dots, S_k$. From the above remark, we know that $S_1, \dots, S_k$ are *i.i.d.* uniform in $(0,t]$, conditioned on the number of points $N(t) = k$. Hence, we can write

$$F_{S_1,\dots,S_k \mid N(t)=k}(t_1, \dots, t_k) = P(\cap_{i=1}^k \{S_i \in (0,t_i]\} \mid \{N(t) = k\}) = \prod_{i=1}^k P(\{S_i \in (0,t_i]\} \mid \{S_i \in (0,t]\}) = \prod_{i=1}^k \frac{t_i}{t} \mathbb{1}_{\{0 < t_i \leqslant t\}}.$$

Therefore, for $0 \leqslant t_1 < \dots < t_k < 1$ and $(h_1, \dots, h_k)$ sufficiently small, we have

$$P\Big(\cap_{i=1}^k \{S_i \in (t_i, t_i + h_i]\} \Big) = \prod_{i=1}^k \frac{h_i}{t}.$$

Since $(S_1, \dots, S_k)$ are conditionally independent given $S \cap A = \{S_1, \dots, S_k\}$, it follows that any permutation $\sigma : [k] \to [k]$, the conditional joint distribution of $(S_{\sigma(1)}, \dots, S_{\sigma(k)})$ is identical to that of $(S_1, \dots, S_k)$ given $S \cap A = \{S_1, \dots, S_k\}$. Further, we observe that the order statistics of $(S_{\sigma(1)}, \dots, S_{\sigma(k)})$ is identical to that of $(S_1, \dots, S_k)$. Therefore, we can write the following equality for the events

$$\cap_{i=1}^k \Big\{ S_{(i)} \in (t_i, t_i + h_i] \Big\} = \cup_{\sigma:[k]\to[k] \text{ permutation}} \cap_{i=1}^k \Big\{ S_{\sigma(i)} \in (t_i, t_i + h_i] \Big\}.$$

The result follows since the number of permutations $\sigma : [k] \to [k]$ is $k!$. □