Lecture-26: Properties of Poisson point processes

1 Laplace functional

Remark 1. Let $\mathcal{X} = \mathbb{R}^d$ be the d-dimensional Euclidean space. For a simple point process $S : \Omega \to \mathcal{X}^{\mathbb{N}}$ all the random points are unique, and hence can be considered as a set of countable points in $\mathcal{X}$. Let $N : \mathcal{B}(\mathcal{X}) \to \mathbb{Z}_+$ be the counting process associated with the simple point process S, then we observe that $dN(x) = 0$ for all $x \notin S$ and $dN(x) = \delta_x \mathbb{1}_{\{x \in S\}}$. Hence, for any function $f : \mathcal{X} \to \mathbb{R}$ and bounded $A \in \mathcal{B}(\mathcal{X})$, we have

$$\int_{x \in A} f(x) dN(x) = \sum_{x \in S \cap A} f(x).$$

Definition 1.1. The **Laplace functional** $\mathcal{L} : \mathbb{R}_+^{\mathcal{X}} \to \mathbb{R}_+$ of a point process $S : \Omega \to \mathcal{X}^{\mathbb{N}}$ and associated counting process $N : \mathcal{B}(\mathcal{X}) \to \mathbb{Z}_+$ is defined for all non-negative Borel measurable function $f : \mathcal{X} \to \mathbb{R}_+$ as

$$\mathcal{L}_S(f) \triangleq \mathbb{E} \exp \left(- \int_{\mathbb{R}^d} f(x) dN(x) \right).$$

Remark 2. For simple function $f = \sum_{i=1}^k t_i \mathbb{1}_{A_i}$, we can write the Laplace functional

$$\mathcal{L}_S(f) = \mathbb{E} \exp \left(- \sum_{i=1}^k t_i \int_{A_i} dN(x) \right) = \mathbb{E} \exp \left(- \sum_{i=1}^k t_i N(A_i) \right),$$

as a function of the vector $(t_1, t_2, \dots, t_k)$, a joint Laplace transform of the random vector $(N(A_1), \dots, N(A_k))$. This way, one can compute all finite dimensional distribution of the counting process N.

Proposition 1.2. *The Laplace functional of a Poisson point process $S : \Omega \to \mathcal{X}^{\mathbb{N}}$ with intensity measure $\Lambda : \mathcal{B}(\mathcal{X}) \to \mathbb{R}_+$, is given by*

$$\mathcal{L}_S(f) = \exp \left(- \int_{\mathcal{X}} (1 - e^{-f(x)}) d\Lambda(x) \right).$$

Proof. For a bounded Borel measurable set $A \in \mathcal{B}(\mathcal{X})$, consider the truncated function $g = f \mathbb{1}_A$. Then,

$$\mathcal{L}_S(g) = \mathbb{E} \exp(- \int_{\mathcal{X}} g(x) dN(x)) = \mathbb{E} \exp(- \int_A f(x) dN(x)).$$

Clearly $dN(x) = \delta_x \mathbb{1}_{\{x \in S\}}$ and hence we can write $\mathcal{L}_S(g) = \mathbb{E} \exp \left(- \sum_{x \in S \cap A} f(x) \right)$. We know that the probability of $N(A) = |S \cap A| = n$ points in set A is given by

$$P \{N(A) = n\} = e^{-\Lambda(A)} \frac{\Lambda(A)^n}{n!}.$$

Given there are n points in set A, the density of n point locations are independent and given by

$$f_{S_1, \dots, S_n \,|\, N(A) = n}(x_1, \dots, x_n) = \prod_{i=1}^n \left(\frac{d\Lambda(x_i)}{\Lambda(A)} \mathbb{1}_{\{x_i \in A\}} \right).$$

Hence, we can write the Laplace functional as

$$\mathcal{L}_S(g) = e^{-\Lambda(A)} \sum_{n \in \mathbb{Z}_+} \frac{\Lambda(A)^n}{n!} \prod_{i=1}^n \int_A e^{-f(x_i)} \frac{d\Lambda(x_i)}{\Lambda(A)} = \exp \left(- \int_{\mathcal{X}} (1 - e^{-g(x)}) d\Lambda(x) \right).$$

Result follows from taking increasing sequences of sets $A_k \uparrow \mathcal{X}$ and monotone convergence theorem. □

1.1 Superposition of point processes

Definition 1.3. Let $S^k : \Omega \to \mathcal{X}^{\mathbb{N}}$ be a simple point process with intensity measures $\Lambda_k : \mathcal{B}(\mathcal{X}) \to \mathbb{R}_+$ and counting process $N_k : \mathcal{B}(\mathcal{X}) \to \mathbb{Z}_+$, for each $k \in \mathbb{N}$. The **superposition** of point processes $(S^k : k \in \mathbb{N})$ is defined as a point process $S \triangleq \cup_k S^k$.

Remark 3. The counting process associated with superposition point process $S : \Omega \to \mathcal{X}^{\mathbb{N}}$ is given by $N : \mathcal{B}(\mathcal{X}) \to \mathbb{Z}_+$ defined by $N \triangleq \sum_k N_k$, and the intensity measure of point process S is given by $\Lambda : \mathcal{B}(\mathcal{X}) \to \mathbb{R}_+$ defined by $\Lambda = \sum_k \Lambda_k$ from monotone convergence theorem.

Remark 4. The superposition process S is simple iff $\sum_k N_k$ is locally finite.

Theorem 1.4. *The superposition of independent Poisson point processes $(S^k : k \in \mathbb{N})$ with intensities $(\Lambda_k : k \in \mathbb{N})$ is a Poisson point process with intensity measure $\sum_k \Lambda_k$ if and only if the latter is a locally finite measure.*

Proof. Consider the superposition $S = \cup_k S^k$ of independent Poisson point processes $S^k \in \mathcal{X}$ with intensity measures Λ_k. We will prove just the sufficiency part this theorem. We assume that $\sum_k \Lambda_k$ is locally finite measure. It is clear that $N(A) = \sum_k N_k(A)$ is finite by locally finite assumption, for all bounded sets $A \in \mathcal{B}(\mathcal{X})$. In particular, we have $dN(x) = \sum_k dN_k(x)$ for all $x \in \mathcal{X}$. From the monotone convergence theorem and the independence of counting processes, we have for a non-negative Borel measurable function $f : \mathcal{X} \to \mathbb{R}$,

$$\mathcal{L}_S(f) = \mathbb{E}\exp\left(-\int_{\mathcal{X}} f(x) \sum_k dN_k(x)\right) = \prod_k \mathcal{L}_{S^k} = \exp\left(-\int_{\mathcal{X}} (1 - e^{-f(x)}) \sum_k \Lambda_k(x)\right).$$

□

1.2 Thinning of point processes

Definition 1.5. Consider a probability **retention function** $p : \mathcal{X} \to [0,1]$ and an independent Bernoulli point retention process $Y : \Omega \to \{0,1\}^{\mathcal{X}}$ such that $\mathbb{E}Y(x) = p(x)$ for all $x \in \mathcal{X}$. The **thinning** of point process $S : \Omega \to \mathcal{X}^{\mathbb{N}}$ with the probability retention function $p : \mathcal{X} \to [0,1]$ is a point process $S^{(p)} : \Omega \to \mathcal{X}^{\mathbb{N}}$ defined by

$$S^{(p)} \triangleq (S_n \in S : Y(S_n) = 1),$$

where $Y(S_n)$ is an independent indicator for the retention of each point S_n and $\mathbb{E}[Y(S_n) \mid S_n] = p(S_n)$.

Theorem 1.6. *The thinning of a Poisson point process $S : \Omega \to \mathcal{X}^{\mathbb{N}}$ of intensity measure $\Lambda : \mathcal{B}(\mathcal{X}) \to \mathbb{R}_+$ with the retention probability function $p : \mathcal{X} \to [0,1]$, yields a Poisson point process $S^{(p)} : \Omega \to \mathcal{X}^{\mathbb{N}}$ of intensity measure $\Lambda^{(p)} : \mathcal{B}(\mathcal{X}) \to \mathbb{R}_+$ defined by*

$$\Lambda^{(p)}(A) \triangleq \int_A p(x) d\Lambda(x), \quad \text{for all bounded } A \in \mathcal{B}(\mathcal{X}).$$

Proof. Let $A \in \mathcal{B}(\mathcal{X})$ be a bounded Boreal measurable set, and let $f : \mathcal{X} \to \mathbb{R}_+$ be a non-negative function. Let $N^{(p)}$ be the associated counting process to the thinned point process $S^{(p)}$. Hence, for any bounded set $A \in \mathcal{B}(\mathcal{X})$, we have $N^{(p)}(A) = \sum_{x \in S \cap A} Y(x)$. That is,

$$dN^{(p)}(x) = \delta_x Y(x) \mathbb{1}_{\{x \in S\}}.$$

Therefore, for any non-negative function $g(x) = f(x)\mathbb{1}_{\{x \in A\}}$, we can write

$$\int_{x \in \mathcal{X}} g(x) dN^{(p)}(x) = \int_{x \in A} f(x) dN^{(p)}(x) = \sum_{x \in S \cap A} f(x) Y(x).$$

We can write the Laplace functional of the thinned point process $S^{(p)}$ for the non-negative function $g(x) = f(x)\mathbb{1}_{\{x \in A\}}$, as

$$\mathcal{L}_{S^{(p)}}(g) = \mathbb{E}\left[\mathbb{E}[\exp\left(-\int_A f(x) dN^p(x)\right) \mid N(A)]\right] = \sum_{n \in \mathbb{Z}_+} P\{N(A) = n\} \prod_{i=1}^{n} \mathbb{E}[^{-f(S_i)Y(S_i)} \Big| S_i \in A].$$

The first equality follows from the definition of Laplace functional and taking nested expectations. Second equality follows from the fact that the distribution of all points of a Poisson point process are *i.i.d.* . Since Y is a Bernoulli process independent of the underlying process S with $\mathbb{E}[Y(S_i)] = p(S_i)$, we get

$$\mathbb{E}[e^{-f(S_i)Y(S_i)} \Big| S_i \in S \cap A] = \mathbb{E}[e^{-f(S_i)}p(S_i) + (1 - p(S_i)) \Big| S_i \in S \cap A].$$

From the distribution $\frac{\Lambda'(x)}{\Lambda(A)}$ for $x \in S \cap A$ for the Poisson point process S, we get

$$\mathcal{L}_{S^{(p)}}(g) = e^{-\Lambda(A)} \sum_{n \in \mathbb{Z}_+} \frac{1}{n!} \left(\int_A (p(x)e^{-f(x)} + (1 - p(x)) d\Lambda(x) \right)^n = \exp\left(- \int_{\mathcal{X}} (1 - e^{-g(x)}) p(x) d\Lambda(x) \right).$$

Result follows from taking increasing sequences of sets $A_k \uparrow \mathcal{X}$ and monotone convergence theorem. □