Lecture-28: Compound Poisson Processes

1 Compound Poisson process

A **compound Poisson process** is a real-valued right-continuous process $Z : \Omega \to \mathbb{R}_+^{\mathbb{R}_+}$ with the following properties.

i_ **Finitely many jumps:** for all $\omega \in \Omega$, sampled path $t \mapsto Z_t(\omega)$ has finitely many jumps in finite intervals,

ii_ **Independent increments:** for all $t, s \geqslant 0; Z_{t+s} - Z_t$ is independent of past $\sigma(Z_u : u \leqslant t)$,

iii_ **Stationary increments:** for all $t, s \geqslant 0$, distribution of $Z_{t+s} - Z_t$ depends only on s and not on t.

For each $\omega \in \Omega$ and $n \in \mathbb{N}$, we can define time and size of nth jump

$$S_0(\omega) = 0, \qquad S_n(\omega) = \inf\{t > S_{n-1} : Z_t(\omega) \neq Z_{S_{n-1}}(\omega)\}$$
$$X_0(\omega) = 0, \qquad X_n(\omega) = Z_{S_n}(\omega) - Z_{S_{n-1}}(\omega).$$

Let $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ be the simple counting process associated with the number of jumps in $(0, t]$ denoted by N_t for all $t \in \mathbb{R}_+$. Then, $\tilde{S}_n$ and X_n are the respectively the arrival instant and the size of the nth jump, and we can write $Z_t = \sum_{i=1}^{N_t} X_i$. Let $\mathcal{F}_s = \sigma(Z_u : u \in (0, s])$ be the collection of historical events until time s associated with the process Z. Clearly, the sequence of jump times $\tilde{S} : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is a sequence of stopping times with respect to the process Z.

Proposition 1.1. *A stochastic process $Z : \Omega \to \mathbb{R}_+^{\mathbb{R}_+}$ is a compound Poisson process iff its jump times form a Poisson process and the jump sizes form an* i.i.d. *random sequence independent of the jump times.*

Proof. We will prove it in two steps.

Implication: We will show that $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ is a Poisson process and $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ is *i.i.d.* .

Counting: It is clear that the simple counting process N can be completely determined by $\sigma(Z_u : u \leqslant t)$, i.e. $N_t \in \mathcal{F}_t$.

Independence: Since $Z_{t+s} - Z_t = \sum_{i=N_t}^{N_{t+s}} X_i$, and the compound Poisson processes have independent increments, it follows that the increment $(N_{t+s} - N_t : s \geqslant 0)$ and $(X_{N_t+j} : j \in \mathbb{N})$ are independent of the past $\mathcal{F}_t$.

Stationarity: Let's assume that step sizes are positive, then we have

$$S_n = \inf\{t > S_{n-1} : Z_t > Z_{S_{n-1}}\}, \qquad \text{and } \{N_{t+s} - N_t = 0\} = \{Z_{t+s} - Z_t = 0\}.$$

From the stationarity of the increments it follows that the probability $P\{N_{t+s} - N_t = 0\}$ is independent of t, and hence $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ is a Poisson process, and that stopping time sequence $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is almost surely finite for each $n \in \mathbb{N}$.

Markovity: The compound Poisson process has the Markov property from stationary and independent increment property. Further, since each sample path $t \mapsto Z_t$ is right continuous, the process satisfies the strong Markov property at each almost sure stopping time.

IID: In particular, X_n is independent of the past $\sigma(Z_u : u \leqslant S_{n-1})$ and identically distributed to X_1 for each $n \in \mathbb{N}$. It follows that the jump sizes $X_1, X_2, \dots$ are *i.i.d.* random variables, independent of arrival instants $S_1, S_2, \dots$.

Superposition: Similar arguments can be used to show for negative jump sizes. For real jump sizes, we can form two independent Poisson processes with negative and positive jumps, and the superposition of these two processes is Poisson.

Converse: Conversely, let $Z_t = \sum_{i=1}^{N_t} X_i$ where N_t is a Poisson process independent of the random *i.i.d.* sequence $X_1, X_2, \ldots$.

Jumps: Since N_t is finite for any finite t, it follows that the compound Poisson process Z has finitely many jumps in finite intervals.

Independence: For any finite $n \in \mathbb{N}$ and finite intervals I_i for $i \in [n]$, we can write $Z(I_i) = \sum_{k=1}^{N(I_i)} X_{ik}$, where X_{ik} denotes the kth jump size in the interval I_i. Since the independent sequence $(N(I_i) : i \in [n])$ and $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ are also mutually independent, it follows that $Z(I_i)$ are independent.

Stationarity: Further, the stationarity of the increments of the compound process is inferred from the distribution of $Z(I_i)$, which is

$$P\{Z(I_i) \leqslant x\} = \sum_{m \in \mathbb{Z}_+} P\{Z(I_i) \leqslant x, N(I_i) = m\} = \sum_{m \in \mathbb{Z}_+} P\left\{\sum_{k=1}^{m} X_{ik} \leqslant x\right\} P\{N(I_i) = m\}.$$

□

Example 1.2. Examples of compound Poisson processes.

- Arrival of customers in a store is a Poison process N. Each customer i spends an *i.i.d.* amount X_i independent of the arrival process.

$$Y_0 = 0, \qquad Y_n = \sum_{i=1}^{n} X_i, i \in [n].$$

  Now define $Z_t \triangleq Y_{N_t}$ as the amount spent by the customers arriving until time $t \in \mathbb{R}_+$. Then $Z : \Omega \to \mathbb{R}_+^{\mathbb{R}_+}$ is a compound Poisson Process.

- Let the time between successive failures of a machine be independent and exponentially distributed. The cost of repair is *i.i.d.* random at each failure. Then the total cost of repair in a certain time t is a compound Poisson Process.