Lecture-04: Random Variable

1 Random Variable

Definition 1.1 (Random variable). Consider a probability space $(\Omega, \mathcal{F}, P)$. A **random variable** $X : \Omega \to \mathbb{R}$ is a real-valued function from the sample space to real numbers, such that for each $x \in \mathbb{R}$ the event

$$A_X(x) \triangleq \{\omega \in \Omega : X(\omega) \leqslant x\} = \{X \leqslant x\} = X^{-1}(-\infty, x] = X^{-1}(B_x) \in \mathcal{F}.$$

We say that the random variable X is $\mathcal{F}$-measurable.

Remark 1. Recall that the set $A_X(x)$ is always a subset of sample space Ω for any mapping $X : \Omega \to \mathbb{R}$, and $A_X(x) \in \mathcal{F}$ is an event when X is a random variable.

Example 1.2 (Constant function). Consider a mapping $X : \Omega \to \{c\} \subseteq \mathbb{R}$ defined on an arbitrary probability space $(\Omega, \mathcal{F}, P)$, such that $X(\omega) = c$ for all outcomes $\omega \in \Omega$. We observe that

$$A_X(x) = X^{-1}(B_x) = \begin{cases} \emptyset, & x < c, \\ \Omega, & x \geqslant c. \end{cases}$$

That is $A_X(x) \in \mathcal{F}$ for all event spaces, and hence X is a random variable and measurable for all event spaces.

Example 1.3 (Indicator function). For an arbitrary probability space $(\Omega, \mathcal{F}, P)$ and an event $A \in \mathcal{F}$, consider the indicator function $\mathbb{1}_A : \Omega \to [0,1]$. Let $x \in \mathbb{R}$, and $B_x = (-\infty, x]$, then it follows that

$$A_X(x) = \mathbb{1}_A^{-1}(B_x) = \begin{cases} \Omega, & x \geqslant 1, \\ A^c, & x \in [0,1), \\ \emptyset, & x < 0. \end{cases}$$

That is, $A_X(x) \in \mathcal{F}$ for all $x \in \mathbb{R}$, and hence the indicator function $\mathbb{1}_A$ is a random variable.

Remark 2. Since any outcome $\omega \in \Omega$ is random, so is the real value $X(\omega)$.

Remark 3. Probability is defined only for events and not for random variables. The events of interest for random variables are the **lower level sets** $A_X(x) = \{\omega : X(\omega) \leqslant x\} = X^{-1}(B_x)$ for any real x.

Remark 4. Consider a probability space $(\Omega, \mathcal{F}, P)$ and a random variable $X : \Omega \to \mathbb{R}$ that is $\mathcal{G}$ measurable for $\mathcal{G} \subseteq \mathcal{F}$. If $\mathcal{G} \subseteq \mathcal{H}$, then X is also $\mathcal{H}$ measurable.

1.1 Distribution function for a random variable

Definition 1.4. For an $\mathcal{F}$ measurable random variable $X : \Omega \to \mathbb{R}$ defined on the probability space $(\Omega, \mathcal{F}, P)$, we can associate a **distribution function** (CDF) $F_X : \mathbb{R} \to [0,1]$ such that for all $x \in \mathbb{R}$,

$$F_X(x) \triangleq P(A_X(x)) = P(\{X \leqslant x\}) = P \circ X^{-1}(-\infty, x] = P \circ X^{-1}(B_x).$$

Example 1.5 (Constant random variable). Let $X : \Omega \to \{c\} \subseteq \mathbb{R}$ be a constant random variable defined on the probability space $(\Omega, \mathcal{F}, P)$. The distribution function is a right-continuous step function at c with step-value unity. That is, $F_X(x) = \mathbb{1}_{[c,\infty)}(x)$. We observe that $P(\{X = c\}) = 1$.

Example 1.6 (Indicator random variable). For an indicator random variable $\mathbb{1}_A : \Omega \to \{0,1\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ and an event $A \in \mathcal{F}$, we have

$$F_X(x) = \begin{cases} 1, & x \geqslant 1, \\ 1 - P(A), & x \in [0,1), \\ 0, & x < 0. \end{cases}$$

Lemma 1.7 (Properties of distribution function). *The distribution function F_X for any random variable X satisfies the following properties.*

1. *The distribution function is monotonically non-decreasing in $x \in \mathbb{R}$.*
2. *The distribution function is right-continuous at all points $x \in \mathbb{R}$.*
3. *The upper limit is $\lim_{x\to\infty} F_X(x) = 1$ and the lower limit is $\lim_{x\to-\infty} F_X(x) = 0$.*

Proof. Let X be a random variable defined on the probability space $(\Omega, \mathcal{F}, P)$.

1. Let $x_1, x_2 \in \mathbb{R}$ such that $x_1 \leqslant x_2$. Then for any $\omega \in A_{x_1}$, we have $X(\omega) \leqslant x_1 \leqslant x_2$, and it follows that $\omega \in A_{x_2}$. This implies that $A_{x_1} \subseteq A_{x_2}$. The result follows from the monotonicity of the probability.
2. For any $x \in \mathbb{R}$, consider any monotonically decreasing sequence $x \in \mathbb{R}^{\mathbb{N}}$ such that $\lim_n x_n = x_0$. It follows that the sequence of events $\left(A_{x_n} = X^{-1}(-\infty, x_n] \in \mathcal{F} : n \in \mathbb{N}\right)$, is monotonically decreasing and hence $\lim_{n\in\mathbb{N}} A_{x_n} = \cap_{n\in\mathbb{N}} A_{x_n} = A_{x_0}$. The right-continuity then follows from the continuity of probability, since

$$F_X(x_0) = P(A_{x_0}) = P(\lim_{n\in\mathbb{N}} A_{x_n}) = \lim_{n\in\mathbb{N}} P(A_{x_n}) = \lim_{x_n \downarrow x} F(x_n).$$

3. Consider a monotonically increasing sequence $x \in \mathbb{R}^{\mathbb{N}}$ such that $\lim_n x_n = \infty$, then $(A_{x_n} \in \mathcal{F} : n \in \mathbb{N})$ is a monotonically increasing sequence of sets and $\lim_n A_{x_n} = \cup_{n\in\mathbb{N}} A_{x_n} = \Omega$. From the continuity of probability, it follows that

$$\lim_{x_n\to\infty} F_X(x_n) = \lim_n P(A_{x_n}) = P(\lim_n A_{x_n}) = P(\Omega) = 1.$$

Similarly, we can take a monotonically decreasing sequence $x \in \mathbb{R}^{\mathbb{N}}$ such that $\lim_n x_n = -\infty$, then $(A_{x_n} \in \mathcal{F} : n \in \mathbb{N})$ is a monotonically decreasing sequence of sets and $\lim_n A_{x_n} = \cap_{n\in\mathbb{N}} A_{x_n} = \emptyset$. From the continuity of probability, it follows that $\lim_{x_n\to-\infty} F_X(x_n) = 0$.

□

Remark 5. If two reals $x_1 < x_2$ then $F_X(x_1) \leqslant F_X(x_2)$ with equality if and only if $P\{(x_1 < X \leqslant x_2\}) = 0$. This follows from the fact that $A_{x_2} = A_{x_1} \cup X^{-1}(x_1, x_2]$.

1.2 Event space generated by a random variable

Definition 1.8 (Event space generated by a random variable). Let $X : \Omega \to \mathbb{R}$ be an $\mathcal{F}$ measurable random variable defined on the probability space $(\Omega, \mathcal{F}, P)$. The smallest event space generated by the events $A_X(x) = X^{-1}(B_x) = X^{-1}(-\infty, x]$ for $x \in \mathbb{R}$ is called the **event space generated** by this random variable X, and denoted by $\sigma(X) \triangleq \sigma(\{A_X(x) : x \in \mathbb{R}\})$.

Remark 6. The event space generated by a random variable is the collection of the inverse of Borel sets, i.e. $\sigma(X) = \{X^{-1}(B) : B \in \mathcal{B}(\mathbb{R})\}$. This follows from the fact that $A_X(x) = X^{-1}(B_x)$ and the inverse map respects countable set operations such as unions, complements, and intersections. That is, if $B \in \mathcal{B}(\mathbb{R}) = \sigma(\{B_x : x \in \mathbb{R}\})$, then $X^{-1}(B) \in \sigma(\{A_X(x) : x \in \mathbb{R}\})$. Similarly, if $A \in \sigma(X) = \sigma(\{A_X(x) : x \in \mathbb{R}\})$, then $A = X^{-1}(B)$ for some $B \in \sigma(\{B_x : x \in \mathbb{R}\})$.

Example 1.9 (Constant random variable). Let $X : \Omega \to \{c\} \subseteq \mathbb{R}$ be a constant random variable defined on the probability space $(\Omega, \mathcal{F}, P)$. Then the smallest event space generated by this random variable is $\sigma(X) = \{\emptyset, \Omega\}$.

Example 1.10 (Indicator random variable). Let $\mathbb{1}_A$ be an indicator random variable defined on the probability space $(\Omega, \mathcal{F}, P)$ and event $A \in \mathcal{F}$, then the smallest event space generated by this random variable is $\sigma(X) = \sigma(\{\emptyset, A^c, \Omega\}) = \{\emptyset, A, A^c, \Omega\}$.

1.3 Discrete random variables

Definition 1.11 (Discrete random variables). If a random variable $X : \Omega \to \mathcal{X} \subseteq \mathbb{R}$ takes countable values on real-line, then it is called a **discrete random variable**. That is, the range of random variable $\mathcal{X}$ is countable, and the random variable is completely specified by the **probability mass function**

$$P_X(x) = P(\{X = x\}), \text{ for all } x \in \mathcal{X}.$$

Example 1.12 (Bernoulli random variable). For the probability space $(\Omega, \mathcal{F}, P)$, the **Bernoulli random variable** is a mapping $X : \Omega \to \{0, 1\}$ and $P_X(1) = p$. We observe that Bernoulli random variable is an indicator for the event $A \triangleq X^{-1}\{1\}$, and $P(A) = p$. Therefore, the distribution function F_X is given by

$$F_X = (1-p)\mathbb{1}_{[0,1)} + \mathbb{1}_{[1,\infty)}.$$

Lemma 1.13. *Any discrete random variable is a linear combination of indicator function over a partition of the sample space.*

Proof. For a discrete random variable $X : \Omega \to \mathcal{X} \subset \mathbb{R}$ on a probability space $(\Omega, \mathcal{F}, P)$, the range $\mathcal{X}$ is countable, and we can define events $E_x \triangleq \{\omega \in \Omega : X(\omega) = x\} \in \mathcal{F}$ for each $x \in \mathcal{X}$. Then the mutually disjoint sequence of events $(E_x \in \mathcal{F} : x \in \mathcal{X})$ partitions the sample space Ω. We can write

$$X(\omega) = \sum_{x \in \mathcal{X}} x \mathbb{1}_{E_x}(\omega).$$

□

Definition 1.14. Any discrete random variable $X : \Omega \to \mathcal{X} \subseteq \mathbb{R}$ defined over a probability space $(\Omega, \mathcal{F}, P)$, with finite range is called a simple random variable.

Example 1.15 (Simple random variables). Let X be a simple random variable, then $X = \sum_{x \in \mathcal{X}} x \mathbb{1}_{A_X(x)}$ where $(A_X(x) = X^{-1}\{x\} \in \mathcal{F} : x \in \mathcal{X})$ is a finite partition of the sample space Ω. Without loss of generality, we can denote $\mathcal{X} = \{x_1, \dots, x_n\}$ where $x_1 \leqslant \dots \leqslant x_n$. Then,

$$X^{-1}(-\infty, x] = \begin{cases} \Omega, & x \geqslant x_n, \\ \cup_{j=1}^{i} A_X(x_j), & x \in [x_i, x_{i+1}), i \in [n-1], \\ \emptyset, & x < x_1. \end{cases}$$

Then the smallest event space generated by the simple random variable X is $\{\cup_{x \in S} A_X(x) : S \subseteq \mathcal{X}\}$.

1.4 Continuous random variables

Definition 1.16. For a continuous random variable X, there exists **density function** $f_X : \mathbb{R} \to [0, \infty)$ such that

$$F_X(x) = \int_{-\infty}^{x} f_X(u)du.$$

Example 1.17 (Gaussian random variable). For a probability space $(\Omega, \mathcal{F}, P)$, **Gaussian random variable** is a continuous random variable $X : \Omega \to \mathbb{R}$ defined by its density function

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),\ x \in \mathbb{R}.$$