Lecture-07: Random Processes

1 Introduction

Remark 1. For an arbitrary index set T, and a real-valued function $x \in \mathbb{R}^T$, the projection operator $\pi_t : \mathbb{R}^T \to \mathbb{R}$ maps $x \in \mathbb{R}^T$ to $\pi_t(x) = x_t$.

Definition 1.1 (Random process). Let $(\Omega, \mathcal{F}, P)$ be a probability space. For an arbitrary index set T and state space $\mathcal{X} \subseteq \mathbb{R}$, a map $X : \Omega \to \mathcal{X}^T$ is called a **random process** if the projections $X_t : \Omega \to \mathcal{X}$ defined by $\omega \mapsto X_t(\omega) \triangleq (\pi_t \circ X)(\omega)$ are random variables on the given probability space.

Definition 1.2. For each outcome $\omega \in \Omega$, we have a function $X(\omega) : T \mapsto \mathcal{X}$ called the **sample path** or the **sample function** of the process X.

Remark 2. A random process X defined on probability space $(\Omega, \mathcal{F}, P)$ with index set T and state space $\mathcal{X} \subseteq \mathbb{R}$, can be thought of as

(a) a map $X : \Omega \times T \to \mathcal{X}$,

(b) a map $X : T \to \mathcal{X}^\Omega$, i.e. a collection of random variables $X_t : \Omega \to \mathcal{X}$ for each time $t \in T$,

(c) a map $X : \Omega \to \mathcal{X}^T$, i.e. a collection of sample functions $X(\omega) : T \to \mathcal{X}$ for each random outcome $\omega \in \Omega$.

1.1 Classification

State space $\mathcal{X}$ can be countable or uncountable, corresponding to discrete or continuous valued process. If the index set $T \subseteq \mathbb{R}$ is countable, the stochastic process is called **discrete**-time stochastic process or random sequence. When the index set T is uncountable, it is called **continuous**-time stochastic process. The index set T doesn't have to be time, if the index set is space, and then the stochastic process is spatial process. When $T = \mathbb{R}^n \times [0, \infty)$, stochastic process X is a spatio-temporal process.

Example 1.3. We list some examples of each such stochastic process.

i_ Discrete random sequence: brand switching, discrete time queues, number of people at bank each day.

ii_ Continuous random sequence: stock prices, currency exchange rates, waiting time in queue of nth arrival, workload at arrivals in time sharing computer systems.

iii_ Discrete random process: counting processes, population sampled at birth-death instants, number of people in queues.

iv_ Continuous random process: water level in a dam, waiting time till service in a queue, location of a mobile node in a network.

1.2 Measurability

For random process $X : \Omega \to \mathcal{X}^T$ defined on the probability space $(\Omega, \mathcal{F}, P)$, the projections $X_t \triangleq \pi_t \circ X$ are $\mathcal{F}$-measurable random variables. Therefore, the set of outcomes $A_{X_t}(x) \triangleq X_t^{-1}(-\infty, x] \in \mathcal{F}$ for all $t \in T$ and $x \in \mathbb{R}$.

Definition 1.4. A random map $X : \Omega \to \mathcal{X}^T$ is called $\mathcal{F}$-**measurable** and hence a random process, if the set of outcomes $A_{X_t}(x) = X_t^{-1}(-\infty, x] \in \mathcal{F}$ for all $t \in T$ and $x \in \mathbb{R}$.

Definition 1.5. The **event space generated by a random process** $X : \Omega \to \mathcal{X}^T$ defined on a probability space $(\Omega, \mathcal{F}, P)$ is given by

$$\sigma(X) \triangleq \sigma(A_{X_t}(x) : t \in T, x \in \mathbb{R}).$$

Definition 1.6. For a random process $X : \Omega \to \mathcal{X}^T$ defined on the probability space $(\Omega, \mathcal{F}, P)$, we define the projection of X onto components $S \subseteq T$ as the random vector $X_S : \Omega \to \mathcal{X}^S$, where $X_S \triangleq (X_s : s \in S)$.

Remark 3. Recall that $\pi_t^{-1}(-\infty, x] = \bigtimes_{s \in T}(-\infty, x_s]$ where $x_s = x$ for $s = t$ and $x_s = \infty$ for all $s \neq t$. The $\mathcal{F}$-measurability of process X implies that for any countable set $S \subseteq T$, we have $A_{X_S}(x_S) \triangleq \cap_{s \in S} A_{X_s}(x_s) \in \mathcal{F}$ for $x_S \in \mathcal{X}^S$.

Remark 4. We can define $A_X(x) \triangleq \cap_{t \in T} A_{X_t}(x_t)$ for any $x \in \mathbb{R}^T$. However, $A_X(x)$ is guaranteed to be an event only when $S \triangleq \{t \in T : \pi_t(x) < \infty\}$ is a countable set. In this case,

$$A_X(x) = \cap_{t \in T} A_{X_t}(x_t) = \cap_{s \in S} A_{X_s}(x_s) = A_{X_S}(x_S) \in \mathcal{F}.$$

Example 1.7 (Bernoulli sequence). Consider a sample space $\{H, T\}^{\mathbb{N}}$. We define a mapping $X : \Omega \to \{0, 1\}^{\mathbb{N}}$ such that $X_n(\omega) = \mathbb{1}_{\{H\}}(\omega_n) = \mathbb{1}_{\{\omega_n = H\}}$. The map X is an $\mathcal{F}$-measurable random sequence, if each $X_n : \Omega \to \{0, 1\}$ is a bi-variate $\mathcal{F}$-measurable random variable on the probability space $(\Omega, \mathcal{F}, P)$. Therefore, the event space $\mathcal{F}$ must contain the event space generated by sequence of events $E \in \mathcal{F}^{\mathbb{N}}$ defined by $E_n \triangleq \{\omega \in \Omega : X_n(\omega) = 1\} = \{\omega \in \Omega : \omega_n = H\} \in \mathcal{F}$ for all $n \in \mathbb{N}$. That is,

$$\sigma(X) = \sigma(E) = \sigma(\{E_n : n \in \mathbb{N}\}).$$

1.3 Distribution

Definition 1.8. For a random process $X : \Omega \to \mathcal{X}^T$ defined on the probability space $(\Omega, \mathcal{F}, P)$, we define a **finite dimensional distribution** $F_{X_S} : \mathbb{R}^S \to [0, 1]$ for a finite $S \subseteq T$ by

$$F_{X_S}(x_S) \triangleq P(A_{X_S}(x_S)), \quad x_S \in \mathbb{R}^S.$$

Example 1.9. Consider a probability space $(\Omega, \mathcal{F}, P)$ defined by the sample space $\Omega = \{H, T\}^{\mathbb{N}}$, the event space $\mathcal{F} \triangleq \sigma(E)$ where $E_n = \{\omega \in \Omega : \omega_n = H\}$ for $n \in \mathbb{N}$, and the probability measure $P : \mathcal{F} \to [0, 1]$ defined by

$$P(\cap_{i \in F} E_i) = p^{|F|}, \text{ for all finite } F \subseteq \mathbb{N}.$$

Let $X : \Omega \to \{0, 1\}^{\mathbb{N}}$ defined as $X_n(\omega) = \mathbb{1}_{E_n}(\omega)$ for all outcomes $\omega \in \Omega$ and $n \in \mathbb{N}$. For this random sequence, we can obtain the finite dimensional distribution $F_{X_S} : \mathbb{R}^S \to [0, 1]$ for any finite $S \subseteq T$ and $x \in \mathbb{R}^S$ in terms of $I_0(x) \triangleq \{i \in S : x_i < 0\}$ and $I_1(x) \triangleq \{i \in S : x_i \in [0, 1)\}$, as

$$F_{X_S}(x) = \begin{cases} 1, & I_0(x) \cup I_1(x) = \emptyset, \\ (1-p)^{|I_1(x)|}, & I_0(x) = \emptyset, I_1(x) \neq \emptyset, \\ 0, & I_0(x) \neq \emptyset. \end{cases} \tag{1}$$

To define a measure on a random process, we can either put a measure on subsets of sample paths $(X(\omega) \in \mathbb{R}^T : \omega \in \Omega)$, or equip the collection of random variables $(X_t \in \mathbb{R}^\Omega : t \in T)$ with a joint measure.

Either way, we are interested in identifying the joint distribution $F : \mathbb{R}^T \to [0,1]$. To this end, for any $x \in \mathbb{R}^T$, we need to know

$$F_X(x) \triangleq P\left(\bigcap_{t\in T}\{\omega \in \Omega : X_t(\omega) \leqslant x_t\}\right) = P(\bigcap_{t\in T} X_t^{-1}(-\infty, x_t]) = P(A_X(x)).$$

First of all, we don't know whether $A_X(x)$ is an event when T is uncountable. Though, we can verify that $A_X(x) \in \mathcal{F}$ for $x \in \mathbb{R}^T$ such that $\{t \in T : x_t < \infty\}$ is countable. Second, even for a simple independent process with countably infinite T, any function of the above form would be zero if x_t is finite for all $t \in T$. Therefore, we only look at the values of $F_X(x)$ for $x \in \mathbb{R}^T$ where $\{t \in T : x_t < \infty\}$ is finite. That is, for any finite set $S \subseteq T$, we focus on the events $A_S(x_S)$ and their probabilities. However, these are precisely the finite dimensional distributions. Set of all finite dimensional distributions of the stochastic process $X : \Omega \to \mathcal{X}^T$ characterizes its distribution completely.

Example 1.10. Consider a probability space $(\Omega, \mathcal{F}, P)$ defined by the sample space $\Omega = \{H, T\}^{\mathbb{N}}$ and the event space $\mathcal{F} \triangleq \sigma(E)$ where $E_n = \{\omega \in \Omega : \omega_n = H\}$ for all $n \in \mathbb{N}$. Let $X : \Omega \to \{0,1\}^{\mathbb{N}}$ defined as $X_n(\omega) = \mathbb{1}_{E_n}(\omega)$ for all outcomes $\omega \in \Omega$ and $n \in \mathbb{N}$. For this random sequence, if we are given the finite dimensional distribution $F_{X_S} : \mathbb{R}^S \to [0,1]$ for any finite $S \subseteq T$ and $x \in \mathbb{R}^S$ in terms of sets $I_0(x) \triangleq \{i \in S : x_i < 0\}$ and $I_1(x) \triangleq \{i \in S : x_i \in [0,1)\}$, as defined in Eq. (**??**). Then, we can find the probability measure $P : \mathcal{F} \to [0,1]$ is given by

$$P(\cap_{i\in F} E_i) = p^{|F|}, \text{ for all finite } F \subseteq \mathbb{N}.$$

1.4 Independence

Definition 1.11. A random process is **independent** if the collection of event spaces $(\sigma(X_t) : t \in T)$ is independent. That is, for all $x_S \in \mathbb{R}^S$, we have

$$F_{X_S}(x_S) = P(\cap_{s\in S}\{X_s \leqslant x_s\}) = \prod_{s\in S} P\{X_s \leqslant x_s\} = \prod_{s\in S} F_{X_s}(x_s).$$

That is, independence of a random process is equivalent to factorization of any finite dimensional distribution function into product of individual marginal distribution functions.

Example 1.12. Consider a probability space $(\Omega, \mathcal{F}, P)$ defined by the sample space $\Omega = \{H, T\}^{\mathbb{N}}$, the event space $\mathcal{F} \triangleq \sigma(E)$ where $E_n = \{\omega \in \Omega : \omega_n = H\}$ for all $n \in \mathbb{N}$, and the probability measure $P : \mathcal{F} \to [0,1]$ defined by

$$P(\cap_{i\in F} E_i) = p^{|F|}, \text{ for all finite } F \subseteq \mathbb{N}.$$

Then, we observe that the random sequence $X : \Omega \to \{0,1\}^{\mathbb{N}}$ defined by $X_n(\omega) \triangleq \mathbb{1}_{E_n}(\omega)$ for all outcomes $\omega \in \Omega$ and $n \in \mathbb{N}$, is independent.

Definition 1.13. Two stochastic processes $X : \Omega \to \mathcal{X}^{T_1}, Y : \Omega \to \mathcal{Y}^{T_2}$ are **independent**, if the corresponding event spaces $\sigma(X), \sigma(Y)$ are independent. That is, for any $x \in \mathbb{R}^{S_1}, y \in \mathbb{R}^{S_2}$ for finite $S_1 \subseteq T_1, S_2 \subseteq T_2$, the events $A_{S_1}(x) \triangleq \cap_{s\in S_1} X_s^{-1}(-\infty, x_s]$ and $B_{S_2}(y) \triangleq \cap_{s\in S_2} Y_s^{-1}(-\infty, y_s]$ are independent. That is, the joint finite dimensional distribution of X and Y factorizes, and

$$P(A_{S_1}(x) \cap B_{S_2}(y)) = P(A_{S_1}(x))P(B_{S_2}(y)) = F_{X_{S_1}}(x)F_{Y_{S_2}}(y), \quad x \in \mathbb{R}^{S_1}, y \in \mathbb{R}^{S_2}.$$