Lecture-15: L^p convergence

1 L^p convergence

Definition 1.1 (Convergence in L^p). Let $p \geqslant 1$, then we say that a random sequence $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ **converges in** L^p to a random variable $X_\infty : \Omega \to \mathbb{R}$, if

$$\lim_n \|X_n - X_\infty\|_p = 0.$$

The convergence in L^p is denoted by $\lim_n X_n = X_\infty$ in L^p.

Remark 1. For $p \in [1, \infty)$, the convergence of a random sequence $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ in L^p to a random variable $X_\infty : \Omega \to \mathbb{R}$ is equivalent to

$$\lim_n \mathbb{E} |X_n - X_\infty|^p = 0.$$

Proposition 1.2 (Convergences L^p implies in probability). *Consider $p \in [1, \infty)$ and a sequence of random variables $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ such that $\lim_n X_n = X_\infty$ in L^p, then $\lim_n X_n = X_\infty$ in probability.*

Proof. Let $\epsilon > 0$, then from the Markov's inequality applied to random variable $|X_n - X|^p$, we have

$$P\{|X_n - X_\infty| > \epsilon\} \leqslant \frac{\mathbb{E}|X_n - X_\infty|^p}{\epsilon}.$$

□

Example 1.3 (Convergence almost surely doesn't imply convergence in L^p). Consider the probability space $([0,1], \mathcal{B}([0,1]), \lambda)$ such that $\lambda([a,b]) = b - a$ for all $0 \leqslant a \leqslant b \leqslant 1$. We define the scaled indicator random variable $X_n : \Omega \to \{0,1\}$ such that

$$X_n(\omega) = 2^n \mathbb{1}_{[0,\frac{1}{n}]}(\omega).$$

We define $N = \{0\}$, and for any $\omega \notin N$, we can find $m \triangleq \lceil \frac{1}{\omega} \rceil$, such that for all $n > m$, we have $X_n(\omega) = 0$. Since $\lambda(N) = 0$, it implies that $\lim_n X_n = 0$ a.s. However, we see that $\mathbb{E}|X_n|^p = \frac{2^{np}}{n}$.

Remark 2. Convergence almost surely implies convergence in probability. Therefore, above example also serves as a counterexample to the fact that convergence in probability doesn't imply convergence in L^p.

Theorem 1.4 (L^2 weak law of large numbers). *Consider a sequence of uncorrelated random variables $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ such that $\mathbb{E}X_n = \mu$ and $\mathrm{Var}(X_n) = \sigma^2$ for all $n \in \mathbb{N}$. Defining the sum $S_n \triangleq \sum_{i=1}^n X_i$ and the n-empirical mean $\bar{X}_n \triangleq \frac{S_n}{n}$, we have $\lim_n \bar{X}_n = \mu$ in L^2 and in probability.*

Proof. From the uncorrelatedness of random sequence X, and linearity of expectation, we get

$$\mathrm{Var}(\bar{X}_n) = \mathbb{E}(\bar{X}_n - \mu)^2 = \frac{1}{n^2}\mathbb{E}(S_n - n\mu)^2 = \frac{\sigma^2}{n}.$$

It follows that $\lim_n \bar{X}_n = \mu$ in L^2. Since the convergence in L^p implies convergence in probability, the result holds. □

Theorem 1.5 (L^1 weak law of large numbers). *Consider an* i.i.d. *random sequence* $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ *defined on a probability space* $(\Omega, \mathcal{F}, P)$ *such that* $\mathbb{E}|X_1| < \infty$ *and* $\mathbb{E}X_1 = \mu$. *Defining the sum* $S_n \triangleq \sum_{i=1}^n X_i$ *and the n-empirical mean* $\bar{X}_n \triangleq \frac{S_n}{n}$, *we have* $\lim_n \bar{X}_n = \mu$ *in probability.*

Example 1.6 (Convergence in L^p doesn't imply almost surely). Consider the probability space $([0,1], \mathcal{B}([0,1]), \lambda)$ such that $\lambda([a,b]) = b - a$ for all $0 \leqslant a \leqslant b \leqslant 1$. For each $k \in \mathbb{N}$, we consider the sequence $S_k = \sum_{i=1}^k i$, and define integer intervals $I_k \triangleq \{S_{k-1}+1, \dots, S_k\}$. Clearly, the intervals $(I_k : k \in \mathbb{N})$ partition the natural numbers, and each $n \in \mathbb{N}$ lies in some I_{k_n}, such that $n = S_{k+n-1} + i_n$ for $i_n \in [k_n]$. Therefore, for each $n \in \mathbb{N}$, we define indicator random variable $X_n : \Omega \to \{0,1\}$ such that

$$X_n(\omega) \triangleq \mathbb{1}_{\left[\frac{i_n-1}{k_n}, \frac{i_n}{k_n}\right]}(\omega).$$

For any $\omega \in [0,1]$, we have $X_n(\omega) = 1$ for infinitely many values since there exist infinitely many (i,k) pairs such that $\frac{(i-1)}{k} \leqslant \omega \leqslant \frac{i}{k}$, and hence $\limsup_n X_n(\omega) = 1$ and hence $\lim_n X_n(\omega) \neq 0$. However, $\lim_n X_n(\omega) = 0$ in L^p, since

$$\mathbb{E}|X_n|^p = \lambda\{X_n(\omega) \neq 0\} = \frac{1}{k_n}.$$

2 L^1 convergence theorems

Theorem 2.1 (Monotone Convergence Theorem). *Consider a non-decreasing non-negative random sequence* $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ *defined on a probability space* $(\Omega, \mathcal{F}, P)$, *such that* $X_n \in L^1$ *for all* $n \in \mathbb{N}$. *Let* $X_\infty(\omega) = \sup_n X_n(\omega)$ *for all* $\omega \in \Omega$, *then* $\mathbb{E}X_\infty = \sup_n \mathbb{E}X_n$.

Proof. From the monotonicity of sequence X and the monotonicity of expectation, we have $\sup_n \mathbb{E}X_n \leqslant \mathbb{E}X_\infty$. Let $\alpha \in (0,1)$ and $Y : \Omega \to \mathbb{R}_+$ a non-negative simple random variable such that $Y \leqslant X_\infty$. We define

$$E_n \triangleq \{\omega \in \Omega : X_n(\omega) \geqslant \alpha Y\} \in \mathcal{F}.$$

From the monotonicity of sequence X, the sequence of events $E \in \mathcal{F}^{\mathbb{N}}$ are monotonically non-decreasing such that $\cup_{n \in \mathbb{N}} E_n = \Omega$. It follows that

$$\alpha \mathbb{E}[Y \mathbb{1}_{E_n}] \leqslant \mathbb{E}[X_n \mathbb{1}_{E_n}] \leqslant \mathbb{E}X_n.$$

We will use the fact that $\lim_n \mathbb{E}[Y \mathbb{1}_{E_n}] = \mathbb{E}[Y]$, then $\alpha \mathbb{E}Y \leqslant \sup_n \mathbb{E}X_n$. Taking supremum over all $\alpha \in (0,1)$ and all simple functions $Y \leqslant X_\infty$, we get $\mathbb{E}X_\infty \leqslant \sup_n \mathbb{E}X_n$. ☐

Theorem 2.2 (Fatou's Lemma). *Consider a non-negative random sequence* $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ *defined on a probability space* $(\Omega, \mathcal{F}, P)$. *Let* $X_\infty(\omega) \triangleq \liminf_n X_n(\omega)$ *for all* $\omega \in \Omega$, *then* $\mathbb{E}X_\infty \leqslant \liminf_n \mathbb{E}X_n$.

Proof. We define $Y_n \triangleq \inf_{k \geqslant n} X_k$ for all $n \in \mathbb{N}$. It follows that $Y : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is a non-negative non-decreasing sequence of random variables, and $X_\infty = \sup_n Y_n = \lim_n Y_n$. Applying monotone convergence theorem to random sequence Y, we get $\mathbb{E}X_\infty = \sup_n \mathbb{E}Y_n$. The result follows from the monotonicity of expectation, and the fact that $Y_n \leqslant X_k$ for all $k \geqslant n$, to get $\mathbb{E}Y_n \leqslant \inf_{k \geqslant n} \mathbb{E}X_k$. ☐

Theorem 2.3 (Dominated Convergence Theorem). *Let* $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ *be a random sequence defined on a probability space* $(\Omega, \mathcal{F}, P)$. *If* $\lim_n X_n = X_\infty$ *a.s. and there exists a* $Y : \Omega \to \mathbb{R}_+$ *such that* $Y \in L^1$ *and* $|X_n| \leqslant Y$ *a.s., then* $\mathbb{E}X_\infty = \lim_n \mathbb{E}X_n$.

Proof. From the hypothesis, we have $Y + X_n \geqslant 0$ a.s. and $Y - X_n \geqslant 0$ a.s. Therefore, from Fatou's Lemma and linearity of expectation, we have

$$\mathbb{E}Y + \mathbb{E}X_\infty \leqslant \liminf_n \mathbb{E}(Y + X_n) = \mathbb{E}Y + \liminf_n \mathbb{E}X_n, \quad \mathbb{E}Y - \mathbb{E}X_\infty \leqslant \liminf_n \mathbb{E}(Y - X_n) = \mathbb{E}Y - \limsup_n \mathbb{E}X_n.$$

Therefore, we have $\limsup_n \mathbb{E}X_n \leqslant \mathbb{E}X_\infty \leqslant \liminf_n \mathbb{E}X_n$, and the result follows. ☐

3 Convergence theorems for conditional means

Proposition 3.1. *Let $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ be a random sequence on the probability space $(\Omega, \mathcal{F}, P)$ such that $\mathbb{E}|X_n| < \infty$ for all $n \in \mathbb{N}$. Let $\mathcal{G}$ and $\mathcal{H}$ be event spaces such that $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$. Then the following theorems hold.*

1. ***Conditional monotone convergence theorem:*** *If $0 \leqslant X_n \leqslant X_{n+1}$ a.s., for all $n \in \mathbb{N}$ and $X_n \to X_\infty$ a.s. for $X_\infty \in L^1$, then $\mathbb{E}[X_n \mid \mathcal{G}] \uparrow \mathbb{E}[X_\infty \mid \mathcal{G}]$ a.s.*
2. ***Conditional Fatou's lemma:*** *If $X_n \geqslant 0$ a.s., for all $n \in \mathbb{N}$, and $\liminf_n X_n \in L^1$, then $\mathbb{E}[\liminf_n X_n \mid \mathcal{G}] \leqslant \liminf_n \mathbb{E}[X_n \mid \mathcal{G}]$ a.s.*
3. ***Conditional dominated convergence theorem:*** *If $|X_n| \leqslant Z$ for all $n \in \mathbb{N}$ and some $Z \in L^1$, and if $X_n \to X_\infty$, a.s., then $\mathbb{E}[X_n \mid \mathcal{G}] \to \mathbb{E}[X_\infty \mid \mathcal{G}]$ a.s. and in L^1.*

Proof. Let $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ be a random sequence on the probability space $(\Omega, \mathcal{F}, P)$ such that $X_n \in L^1$ for all $n \in \mathbb{N}$.

1. **Conditional monotone-convergence theorem:** By monotonicity, we have $\mathbb{E}[X_n \mid \mathcal{G}] \uparrow Y$ a.s. where $Y : \Omega \to \mathbb{R}_+$ is $\mathcal{G}$ measurable. The monotone convergence theorem implies that, for each $G \in \mathcal{G}$,

$$\mathbb{E}[Y \mathbb{1}_G] = \lim_n \mathbb{E}[\mathbb{1}_G \mathbb{E}[X_n \mid \mathcal{G}]] = \lim_n \mathbb{E}[\mathbb{1}_G X_n] = \mathbb{E}[\mathbb{1}_G X_\infty].$$

2. **Conditional Fatou's lemma:** Defining $Y_n \triangleq \inf_{k \geqslant n} X_k$, we get $Y_n \uparrow Y_\infty = \liminf_k X_k$. By monotonicity,

$$\mathbb{E}[Y_n \mid \mathcal{G}] \leqslant \inf_{k \geqslant n} \mathbb{E}[X_k \mid \mathcal{G}] \text{ a.s..}$$

The conditional monotone-convergence theorem implies that

$$\mathbb{E}[Y_\infty \mid \mathcal{G}] = \lim_{n \in \mathbb{N}} \mathbb{E}[Y_n \mid \mathcal{G}] \leqslant \liminf_n \mathbb{E}[X_n \mid \mathcal{G}] \text{ a.s..}$$

3. **Conditional dominated-convergence theorem:** By the conditional Fatou's lemma, we have

$$\mathbb{E}[Z + X_\infty \mid \mathcal{G}] \leqslant \liminf_n \mathbb{E}[Z + X_n \mid \mathcal{G}] \text{ a.s. ,} \qquad \mathbb{E}[Z - X_\infty \mid \mathcal{G}] \leqslant \liminf_n \mathbb{E}[Z - X_n \mid \mathcal{G}] \text{ a.s. ,}$$

and the a.s.-statement follows.

□

4 Uniform integrability

Definition 4.1 (uniform integrability). A family $(X_t \in L^1 : t \in T)$ of random variables indexed by T is **uniformly integrable** if

$$\lim_{a \to \infty} \sup_{t \in T} \mathbb{E}[|X_t| \mathbb{1}_{\{|X_t| > a\}}] = 0.$$

Example 4.2 (Single element family). If $|T| = 1$, then the family is uniformly integrable, since $X_1 \in L^1$ and $\lim_a \mathbb{E}[|X_1| \mathbb{1}_{\{|X_t| > a\}}] = 0$. This is due to the fact that $(X_n \triangleq |X| \mathbb{1}_{\{|X| \leqslant n\}} : n \in \mathbb{N})$ is a sequence of increasing random variables $\lim_n X_n = X$. From monotone convergence theorem, we get $\lim_n \mathbb{E}|X_n| = \mathbb{E} \lim_n |X_n|$. Therefore,

$$\lim_a \mathbb{E}[|X| \mathbb{1}_{\{|X| > a\}}] = \mathbb{E}|X| - \lim_a \mathbb{E}[|X| \mathbb{1}_{\{|X| \leqslant a\}}] = 0.$$

Proposition 4.3. *Let $X \in L^p$ and $(A_n : n \in \mathbb{N}) \subset \mathcal{F}$ be a sequence of events such that $\lim_n P(A_n) = 0$, then*

$$\lim_n \left\| |X| \mathbb{1}_{A_n} \right\|_p = 0.$$

Example 4.4 (Dominated family). If there exists $Y \in L^1$ such that $\sup_{t \in T} |X_t| \leqslant |Y|$, then the family of random variables $(X_t : t \in T)$ is uniformly integrable. This is due to the fact that

$$\sup_{t \in T} \mathbb{E}[|X| \mathbb{1}_{\{|X|>a\}}] \leqslant \mathbb{E}[|Y| \mathbb{1}_{\{|Y|>a\}}].$$

Example 4.5 (Finite family). then the family of random variables $(X_t : t \in T)$ is uniformly integrable. This is due to the fact that $\sup_{t \in T} |X_t| \leqslant \sum_{t \in T} |X_t| \in L^1$.

Theorem 4.6 (Convergence in probability with uniform integrability implies convergence in L^p). *Consider a sequence of random variables $(X_n : n \in \mathbb{N}) \subset L^p$ for $p \geqslant 1$. Then the following are equivalent.*

(a) The sequence $(X_n : n \in \mathbb{N})$ converges in L^p, i.e. $\lim_n \mathbb{E}|X_n - X|^p = 0$.

(b) The sequence $(X_n : n \in \mathbb{N})$ is Cauchy in L^p, i.e. $\lim_{m,n \to \infty} \mathbb{E}|X_n - X_m|^p = 0$.

(c) $\lim_n X_n = X$ in probability and the sequence $(|X_n|^p : n \in \mathbb{N})$ is uniformly integrable.

Proof. For a random sequence $(X_n : n \in \mathbb{N})$ in L^p, we will show that $(a) \implies (b) \implies (c) \implies (a)$.

$(a) \implies (b)$: We assume the sequence $(X_n : n \in \mathbb{N})$ converges in L^p. Then, from Minkowski's inequality, we can write

$$(\mathbb{E}|X_n - X_m|^p)^{\frac{1}{p}} \leqslant (\mathbb{E}|X_n - X|^p)^{\frac{1}{p}} + (\mathbb{E}|X_m - X|^p)^{\frac{1}{p}}.$$

$(b) \implies (c)$: We assume that the sequence $(X_n : n \in \mathbb{N})$ is Cauchy in L^p, i.e. $\lim_{m,n \to \infty} \mathbb{E}|X_n - X_m|^p = 0$. Let $\epsilon > 0$, then for each $n \in \mathbb{N}$, there exists N_ϵ such that for all $n, m \geqslant N_\epsilon$

$$\mathbb{E}|X_n - X_m|^p \leqslant \frac{\epsilon}{2}.$$

Let $A_a = \{\omega \in A : |X_n| > a\}$. Then, using triangle inequality and the fact that $\mathbb{1}_{A_a} \leqslant 1$, from the linearity and monotonicity of expectation, we can write for $n \geqslant N_\epsilon$

$$(\mathbb{E}[|X_n|^p \mathbb{1}_{\{|X_n|>a\}}])^{\frac{1}{p}} \leqslant (\mathbb{E}[|X_{N_\epsilon}|^p \mathbb{1}_{A_a}])^{\frac{1}{p}} + (\mathbb{E}[|X_n - X_{N_\epsilon}|^p])^{\frac{1}{p}} \leqslant (\mathbb{E}[|X_{N_\epsilon}|^p \mathbb{1}_{A_a}])^{\frac{1}{p}} + \frac{\epsilon}{2}.$$

Therefore, we can write $\sup_n \mathbb{E}[|X_n|^p \mathbb{1}_{\{|X_n|>a\}}] \leqslant \sup_{m \leqslant N_\epsilon} \mathbb{E}[|X_m|^p \mathbb{1}_{A_a}] + \frac{\epsilon}{2}$. Since $(|X_n|^p : n \leqslant N_\epsilon)$ is finite family of random variables in L^1, it is uniformly integrable. Therefore, there exists $a_\epsilon \in \mathbb{R}_+$ such that $\sup_{m \leqslant N_\epsilon} (\mathbb{E}[|X_m|^p \mathbb{1}_{A_a}])^{\frac{1}{p}} < \frac{\epsilon}{2}$. Taking $a' = \max\{a, a_\epsilon\}$, we get $\sup_n (\mathbb{E}[|X_n|^p \mathbb{1}_{\{|X_n|>a'\}}])^{\frac{1}{p}} \leqslant \epsilon$. Since the choice of ϵ was arbitrary, it follows that

$$\lim_{a \to \infty} \sup_n (\mathbb{E}[|X_n|^p \mathbb{1}_{\{|X_n|>a'\}}])^{\frac{1}{p}} = 0.$$

The convergence in probability follows from the Markov inequality, i.e.

$$P\{|X_n - X_m|^p > \epsilon\} \leqslant \frac{1}{\epsilon} \mathbb{E}|X_n - X_m|^p.$$

$(c) \implies (a)$: Since the sequence $(X_n : n \in \mathbb{N})$ is convergent in probability to a random variable X, there exists a subsequence $(n_k : k \in \mathbb{N}) \subset \mathbb{N}$ such that $\lim_k X_{n_k} = X$ a.s. Since $(|X_n|^p : n \in \mathbb{N})$ is a family of uniformly integrable sequence, by Fatou's Lemma

$$\mathbb{E}\,|X|^p \leqslant \liminf_k \mathbb{E}\left|X_{n_k}\right|^p \leqslant \sup_n \mathbb{E}\,|X_n|^p < \infty.$$

Therefore, $X \in L^1$, and we define $A_n(\epsilon) = \{|X_n - X| > \epsilon\}$ for any $\epsilon > 0$. From Minkowski's inequality, we get

$$\|X_n - X\|_p \leqslant \left\|(X_n - X)\mathbb{1}_{\{|X_n - X|^p \leqslant \epsilon\}}\right\|_p + \left\|X_n \mathbb{1}_{A_n(\epsilon)}\right\|_p + \left\|X \mathbb{1}_{A_n(\epsilon)}\right\|_p.$$

We can check that $\left\|(X_n - X)\mathbb{1}_{A_n^c(\epsilon)}\right\|_p \leqslant \epsilon$. Further, since $\lim_n X_n = X$ in probability, $(A_n : n \in \mathbb{N}) \subset \mathcal{F}$ is decreasing sequence of events, and since $X_n, X \in L^1$, we have $\lim_n \left\|X_n \mathbb{1}_{A_n(\epsilon)}\right\| = \lim_n \left\|X \mathbb{1}_{A_n(\epsilon)}\right\| = 0.$

□