

Lecture-16: Weak convergence

1 Convergence in distribution

Definition 1.1 (convergence in distribution). A random sequence $X : \Omega \rightarrow \mathbb{R}^{\mathbb{N}}$ defined on a probability space (Ω, \mathcal{F}, P) **converges in distribution** to a random variable $X_{\infty} : \Omega' \rightarrow \mathbb{R}$ defined on a probability space $(\Omega', \mathcal{F}', P')$ if $\lim_n F_{X_n}(x) = F_{X_{\infty}}(x)$ at all continuity points x of $F_{X_{\infty}}$. Convergence in distribution is denoted by $\lim_n X_n = X_{\infty}$ in distribution.

Proposition 1.2. Consider a random sequence $X : \Omega \rightarrow \mathbb{R}^{\mathbb{N}}$ defined on a probability space (Ω, \mathcal{F}, P) and a random variable $X_{\infty} : \Omega' \rightarrow \mathbb{R}$ defined on another probability space $(\Omega', \mathcal{F}', P')$. Then the following statements are equivalent.

- (a) $\lim_n X_n = X_{\infty}$ in distribution.
- (b) $\lim_n \mathbb{E}[g(X_n)] = \mathbb{E}[g(X_{\infty})]$ for any bounded continuous Borel measurable function $g : \mathbb{R} \rightarrow \mathbb{R}$.
- (c) Characteristic functions converge point-wise, i.e. $\lim_n \Phi_{X_n}(u) = \Phi_{X_{\infty}}(u)$ for each $u \in \mathbb{R}$.

Proof. Let $X : \Omega \rightarrow \mathbb{R}^{\mathbb{N}}$ be a sequence of random variables and let $X_{\infty} : \Omega' \rightarrow \mathbb{R}$ be a random variable. We will show that (a) \implies (b) \implies (c) \implies (a).

- (a) \implies (b): Applying the bounded convergence theorem to any bounded continuous Borel measurable function $g : \mathbb{R} \rightarrow \mathbb{R}$, we have $\lim_n \int_{x \in \mathbb{R}} g(x) dF_{X_n}(x) = \int_{x \in \mathbb{R}} g(x) \lim_n dF_{X_n}(x)$.
- (b) \implies (c): Taking $g(x) = e^{jux}$, we get the result.
- (c) \implies (a): The proof of this part is technical and is omitted.

□

Example 1.3 (Convergence in distribution but not in probability). Consider a sequence of non-degenerate continuous *i.i.d.* random variables $X : \Omega \rightarrow \mathbb{R}^{\mathbb{N}}$ and independent random variable $Y : \Omega' \rightarrow \mathbb{R}$, all with the common distribution F_Y . Then $F_{X_n} = F_Y$ for all $n \in \mathbb{N}$, and hence $\lim_n X_n = Y$ in distribution. If the common distribution F_Y is zero mean Gaussian with variance σ^2 , then $X_n - Y$ is zero mean Gaussian with variance $2\sigma^2$. Therefore, for $\epsilon < \sigma\sqrt{\pi}$ and all $n \in \mathbb{N}$

$$P\{|X_n - Y| \leq \epsilon\} = \frac{1}{\sqrt{4\pi\sigma^2}} \int_{z \in [-\epsilon, \epsilon]} e^{-z^2/4\sigma^2} dz \leq \frac{\epsilon}{\sigma\sqrt{\pi}} < 1.$$

It follows that $P\{|X_n - Y| > \epsilon\} \geq 1 - \frac{\epsilon}{\sigma\sqrt{\pi}}$ for all $n \in \mathbb{N}$, and hence $\lim X_n \neq Y$ in probability.

Lemma 1.4 (Convergence in probability implies in distribution). Consider a sequence $X : \Omega \rightarrow \mathbb{R}^{\mathbb{N}}$ of random variables and a random variable $X_{\infty} : \Omega \rightarrow \mathbb{R}$ defined on a probability space (Ω, \mathcal{F}, P) , such that $\lim_n X_n = X_{\infty}$ in probability, then $\lim_n X_n = X_{\infty}$ in distribution.

Proof. We will show that all continuity points x of $F_{X_{\infty}}$, we have $\lim_{n \rightarrow \infty} F_{X_n}(x) = F_{X_{\infty}}(x)$. Fix $\epsilon > 0$. Since x is a continuity point of non-decreasing function $F_{X_{\infty}}$, choose $\delta > 0$ such that $F_{X_{\infty}}(x + \delta) - F_{X_{\infty}}(x - \delta) < \epsilon$. Therefore, it suffices to show that

$$F_{X_{\infty}}(x - \delta) \leq \liminf_{n \rightarrow \infty} F_{X_n}(x) \leq \limsup_{n \rightarrow \infty} F_{X_n}(x) \leq F_{X_{\infty}}(x + \delta).$$

For the chosen $\delta > 0$, we consider the event $A_n(\delta) \triangleq \{\omega \in \Omega : |X_n(\omega) - X_\infty(\omega)| > \delta\} = \{X_n \notin [X_\infty - \delta, X_\infty + \delta]\} \in \mathcal{F}$, and define events $A_{X_n}(x) \triangleq \{X_n \leq x\}$ and $A_{X_\infty}(x) \triangleq \{X_\infty \leq x\}$. Then, we can write

$$\begin{aligned} A_{X_n}(x) \cap A_{X_\infty}(x + \delta) &\subseteq A_{X_\infty}(x + \delta), & A_{X_n}(x) \cap A_{X_\infty}^c(x + \delta) &\subseteq A_n(\delta), \\ A_{X_\infty}(x - \delta) \cap A_{X_n}(x) &\subseteq A_{X_n}(x), & A_{X_\infty}(x - \delta) \cap A_{X_n}^c(x) &\subseteq A_n(\delta). \end{aligned}$$

From the above set relations, law of total probability, and union bound, we have

$$F_{X_\infty}(x - \delta) - P(A_n(\delta)) \leq F_{X_n}(x) \leq F_{X_\infty}(x + \delta) + P(A_n(\delta)).$$

From the convergence in probability, we have $\lim_n P(A_n(\delta)) = 0$, and the result follows. \square

Theorem 1.5 (Central Limit Theorem). Consider an i.i.d. random sequence $X : \Omega \rightarrow \mathbb{R}^{\mathbb{N}}$ defined on a probability space (Ω, \mathcal{F}, P) , with $\mathbb{E}X_n = \mu$ and $\text{Var}(X_n) = \sigma^2$ for all $n \in \mathbb{N}$. We define the n -sum as $S_n \triangleq \sum_{i=1}^n X_i$ and consider a standard normal random variable $Y : \Omega \rightarrow \mathbb{R}$ with density function $f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$ for all $y \in \mathbb{R}$. Then,

$$\lim_n \frac{S_n - n\mu}{\sigma\sqrt{n}} = Y \text{ in distribution.}$$

Proof. The classical proof is using the characteristic functions. Let $Z_i \triangleq \frac{X_i - \mu}{\sigma}$ for all $i \in \mathbb{N}$, then the shifted and scaled n -sum is given by $\frac{S_n - n\mu}{\sigma\sqrt{n}} = \frac{1}{\sqrt{n}} \sum_{i=1}^n Z_i$. We use the third equivalence in Proposition 1.2 to show that the characteristic function of converges to the characteristic function of the standard normal. We define the characteristic functions

$$\Phi_n(u) \triangleq \mathbb{E} \exp\left(ju \frac{(S_n - n\mu)}{\sigma\sqrt{n}}\right), \quad \Phi_{Z_i}(u) \triangleq \mathbb{E} \exp(juZ_i), \quad \Phi_Y(u) \triangleq \mathbb{E} \exp(juY).$$

We can compute the characteristic function of the standard normal as

$$\Phi_Y(u) = \frac{1}{\sqrt{2\pi}} \int_{y \in \mathbb{R}} e^{-\frac{y^2}{2}} \exp\left(-\frac{(y - ju)^2}{2}\right) dy = e^{-\frac{u^2}{2}}.$$

Since the random sequence $Z : \Omega \rightarrow \mathbb{R}^{\mathbb{N}}$ is a zero mean i.i.d. sequence, it follows that $\Phi_{Z_1}^{(1)}(0) = j\mathbb{E}Z_1 = 0$ and $\Phi_{Z_1}^{(2)}(0) = j^2\mathbb{E}Z_1^2 = -1$. Using the Taylor expansion of the characteristic function Φ_{Z_1} , we have

$$\Phi_n(u) = \prod_{i=1}^n \mathbb{E} \exp\left(ju \frac{(X_i - \mu)}{\sigma\sqrt{n}}\right) = \left[\Phi_{Z_1}\left(\frac{u}{\sqrt{n}}\right)\right]^n = \left[1 - \frac{u^2}{2n} + o\left(\frac{u^2}{n}\right)\right]^n.$$

For any $u \in \mathbb{R}$, taking limit $n \in \mathbb{N}$, we get the result. \square

2 Strong law of large numbers

Definition 2.1. For a random sequence $X : \Omega \rightarrow \mathbb{R}^{\mathbb{N}}$ defined on a probability space (Ω, \mathcal{F}, P) with bounded mean $\mathbb{E}|X_n| < \infty$ for all $n \in \mathbb{N}$, we define the n -sum as $S_n \triangleq \sum_{i=1}^n X_i$ and the empirical n -mean $\frac{S_n}{n}$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we define event

$$E_n \triangleq \{\omega \in \Omega : |S_n - \mathbb{E}S_n| > n\epsilon\} \in \mathcal{F}.$$

Theorem 2.2 (L^4 strong law of large numbers). Let $X : \Omega \rightarrow \mathbb{R}^{\mathbb{N}}$ be a sequence of independent random variables defined on probability space (Ω, \mathcal{F}, P) with bounded mean $\mathbb{E}X_n$ for each $n \in \mathbb{N}$ and uniformly bounded fourth central moment $\sup_{n \in \mathbb{N}} \mathbb{E}(X_n - \mathbb{E}X_n)^4 \leq B < \infty$. Then, the empirical n -mean converges to $\lim_n \frac{\mathbb{E}S_n}{n}$ almost surely.

Proof. Recall that $\mathbb{E}(S_n - \mathbb{E}S_n)^4 = \mathbb{E}(\sum_{i=1}^n (X_i - \mathbb{E}X_i))^4 = \sum_{i=1}^n \mathbb{E}(X_i - \mathbb{E}X_i)^4 + 3 \sum_{i=1}^n \sum_{j \neq i} \mathbb{E}(X_i - \mathbb{E}X_i)^2 \mathbb{E}(X_j - \mathbb{E}X_j)^2$. Recall that when the fourth moment is bounded, then so is second moment. Hence, $\sup_{i \in \mathbb{N}} \mathbb{E}(X_i - \mathbb{E}X_i)^2 \leq C$ for some $C \in \mathbb{R}_+$. Therefore, from the Markov's inequality, we have

$$P(E_n) \leq \frac{\mathbb{E}(S_n - \mathbb{E}S_n)^4}{n^4 \epsilon^4} \leq \frac{nB + 3n(n-1)C^2}{n^4 \epsilon^4}.$$

It follows that the $\sum_{n \in \mathbb{N}} P(E_n) < \infty$, and hence by Borel Canteli Lemma, we have

$$P\{E_n^c \text{ for all but finitely many } n\} = 1.$$

Since, the choice of ϵ was arbitrary, the result follows. \square

Theorem 2.3 (L^2 strong law of large numbers). *Let $X : \Omega \rightarrow \mathbb{R}^{\mathbb{N}}$ be a sequence of pair-wise uncorrelated random variables defined on a probability space (Ω, \mathcal{F}, P) with bounded mean $\mathbb{E}X_n$ for all $n \in \mathbb{N}$ and uniformly bounded variance $\sup_{n \in \mathbb{N}} \text{Var}(X_n) \leq B < \infty$. Then, the empirical n -mean converges to $\lim_n \frac{\mathbb{E}S_n}{n}$ almost surely.*

Proof. For each $n \in \mathbb{N}$, we define events $F_n \triangleq E_{n^2}$, and

$$G_n \triangleq \left\{ \max_{n^2 \leq k < (n+1)^2} |S_k - S_{n^2} - \mathbb{E}(S_k - S_{n^2})| > n^2 \epsilon \right\} = \bigcup_{n^2 \leq k < (n+1)^2} \left\{ \omega \in \Omega : \left| \sum_{i=n^2}^k (X_i - \mathbb{E}X_i) \right| > n^2 \epsilon \right\}.$$

From the Markov's inequality and union bound, we have

$$P(F_n) \leq \frac{\sum_{i=1}^{n^2} \text{Var}(X_i)}{n^4 \epsilon^2} \leq \frac{B}{n^2 \epsilon^2}, \quad P(G_n) \leq \sum_{k=n^2}^{(n+1)^2-1} \frac{(k - n^2 + 1)B}{n^4 \epsilon^2} \leq \frac{(2n+1)^2 B}{n^4 \epsilon^2}.$$

Therefore, $\sum_{n \in \mathbb{N}} P(F_n) < \infty$ and $\sum_{n \in \mathbb{N}} P(G_n) < \infty$, and hence by Borel Canteli Lemma, we have

$$\lim_n \frac{S_{n^2} - \mathbb{E}S_{n^2}}{n^2} = \lim_n \max_{n^2 \leq k < (n+1)^2-1} \frac{S_k - S_{n^2} - \mathbb{E}(S_k - S_{n^2})}{n^2} = 0 \text{ a.s.}$$

The result follows from the fact that for any $k \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $k \in \{n^2, \dots, (n+1)^2 - 1\}$ and hence

$$\frac{|S_k - \mathbb{E}S_k|}{k} \leq \left(\frac{|S_{n^2} - \mathbb{E}S_{n^2}|}{n^2} + \frac{|S_k - S_{n^2} - \mathbb{E}(S_k - S_{n^2})|}{n^2} \right).$$

\square

Theorem 2.4 (L^1 strong law of large numbers). *Let $X : \Omega \rightarrow \mathbb{R}^{\mathbb{N}}$ be a random sequence defined on a probability space (Ω, \mathcal{F}, P) such that $\sup_{n \in \mathbb{N}} \mathbb{E}|X_n| \leq B < \infty$. Then, the empirical n -mean converges to $\lim_n \frac{\mathbb{E}S_n}{n}$ almost surely.*