Lecture-17: Tractable Random Processes

1 Examples of Tractable Stochastic Processes

Recall that a random process $X : \Omega \to \mathcal{X}^T$ defined on the probability space $(\Omega, \mathcal{F}, P)$ with index set T and state space $\mathcal{X} \subseteq \mathbb{R}$, is completely characterized by its finite dimensional distributions $F_{X_S} : \mathbb{R}^S \to [0,1]$ for all finite $S \subseteq T$, where

$$F_{X_S}(x_S) \triangleq P(A_{X_S}(x_S)) = P(\cap_{s \in S} X_s^{-1}(-\infty, x_s]), \quad x_S \in \mathbb{R}^S.$$

Simpler characterizations of a stochastic process X are in terms of its moments. That is, the first moment such as mean, and the second moment such as correlations and covariance functions.

$$m_X(t) \triangleq \mathbb{E}X_t, \qquad R_X(t,s) \triangleq \mathbb{E}X_t X_s, \qquad C_X(t,s) \triangleq \mathbb{E}(X_t - m_X(t))(X_s - m_X(s)).$$

In general, it is very difficult to characterize a stochastic process completely in terms of its finite dimensional distribution. However, we have listed few analytically tractable examples below, where we can completely characterize the stochastic process.

1.1 Independent and identically distributed (*i.i.d.*) processes

Definition 1.1 (*i.i.d.* process). A random process $X : \Omega \to \mathcal{X}^T$ is an **independent and identically distributed** (*i.i.d.*) random process with the common distribution $F : \mathbb{R} \to [0,1]$, if for any finite $S \subseteq T$ and a real vector $x_S \in \mathbb{R}^S$ we can write the finite dimensional distribution for this process as

$$F_{X_S}(x_S) = P\left(\cap_{s \in S} \{X_s(\omega) \leqslant x_s\}\right) = \prod_{s \in S} F(x_s).$$

Remark 1. It's easy to verify that the first and the second moments are independent of time indices. That is, if $0 \in T$ then $X_t = X_0$ in distribution, and we have

$$m_X = \mathbb{E}X_0, \qquad R_X(t,s) = (\mathbb{E}X_0^2)\mathbb{1}_{\{t=s\}} + m_X^2 \mathbb{1}_{\{t \neq s\}}, \qquad C_X(t,s) = \mathrm{Var}(X_0)\mathbb{1}_{\{t=s\}}.$$

1.2 Stationary processes

Definition 1.2 (Stationary process). We consider the index set $T \subseteq \mathbb{R}$ that is closed under addition and subtraction. A stochastic process $X : \Omega \to \mathcal{X}^T$ is **stationary** if all finite dimensional distributions are shift invariant. That is, for any finite $S \subseteq T$ and $t \in T$, we have

$$F_{X_S}(x_S) = P(\cap_{s \in S} \{X_s(\omega) \leqslant x_s\}) = P(\cap_{s \in S} \{X_{s+t}(\omega) \leqslant x_s\}) = F_{X_{t+S}}(x_S).$$

Remark 2. That is, for any finite $n \in \mathbb{N}$ and $t \in \mathbb{R}$, the random vectors $(X_{s_1}, \dots, X_{s_n})$ and $(X_{s_1+t}, \dots, X_{s_1+t})$ have the identical joint distribution for all $s_1 \leqslant \dots \leqslant s_n$.

Lemma 1.3. *Any* i.i.d. *process with index set $T \subseteq \mathbb{R}$ is stationary.*

Proof. Let $X : \Omega \to \mathcal{X}^T$ be an *i.i.d.* random process, where $T \subseteq \mathbb{R}$. Then, for any finite index subset $S \subseteq T, t \in T$ and $x_S \in \mathbb{R}^S$, we can write

$$F_{X_S}(x_S) = P(\bigcap_{s\in S}\{X_s \leqslant x_s\}) = \prod_{s\in S} P\circ X_s^{-1}(-\infty,x_s] = \prod_{s\in S} P\circ X_{t+s}^{-1}(-\infty,x_s] = P(\bigcap_{s\in t+S}\{X_s \leqslant x_s\}) = F_{X_{t+S}}(x_S).$$

First equality follows from the definition, the second from the independence of process X, the third from the identical distribution for the process X. In particular, we have shown that process X is also stationary. □

Remark 3. For a stationary stochastic process, all the existing moments are shift invariant when they exist.

Definition 1.4. A **second order** stochastic process X has finite auto-correlation $R_X(t,t) < \infty$ for all $t \in T$.

Remark 4. This implies $R_X(t_1,t_2) < \infty$ by Cauchy-Schwartz inequality, and hence the mean, auto-correlation, and the auto-covariance functions are well defined and finite.

Remark 5. For a stationary process X, we have $X_t = X_0$ and $(X_t,X_s) = (X_{t-s},X_0)$ in distribution. Therefore, for a second order stationary process X, we have

$$m_X = \mathbb{E}X_0, \qquad R_X(t,s) = R_X(t-s,0) = \mathbb{E}X_{t-s}X_0, \qquad C_X(t-s,0) = R_X(t-s,0) - m_X^2.$$

Definition 1.5. A random process X is **wide sense stationary** if

1. $m_X(t) = m_X(t+s)$ for all $s,t \in T$, and
2. $R_X(t,s) = R_x(t+u,s+u)$ for all $s,t,u \in T$.

Remark 6. It follows that a second order stationary stochastic process X, is wide sense stationary. A second order wide sense stationary process is not necessarily stationary. We can similarly define joint stationarity and joint wide sense stationarity for two stochastic processes X and Y.

Example 1.6 (Gaussian process). Let $X : \Omega \to \mathbb{R}^{\mathbb{R}}$ be a continuous-time Gaussian process, defined by its finite dimensional distributions. In particular, for any finite $S \subset \mathbb{R}$, column vector $x_S \in \mathbb{R}^S$, mean vector $\mu_S \triangleq \mathbb{E}X_S$, and the covariance matrix $C_S \triangleq \mathbb{E}X_S X_S^T$, the finite-dimensional density is given by

$$f_{X_S}(x_S) = \frac{1}{(2\pi)^{|S|/2}\sqrt{\det(C_S)}}\exp\left(-\frac{1}{2}(x_S-\mu_S)^T C_S^{-1}(x_S-\mu_S)\right).$$

Theorem 1.7. *A wide sense stationary Gaussian process is stationary.*

Proof. For Gaussian random processes, first and the second moment suffice to get any finite dimensional distribution. Let X be a wide sense stationary Gaussian process and let $S \subseteq \mathbb{R}$ be finite. From the wide sense stationarity of X, we have $\mathbb{E}X_S = \mu\mathbf{1}_S$ and

$$\mathbb{E}(X_s-\mu)(X_u-\mu) = C_{s-u}, \text{ for all } s,u \in S.$$

This means that $C_S = C_{t+S}$, and the result follows.

1.3 Random Walk

Definition 1.8. Let $X : \Omega \to \mathcal{X}^{\mathbb{N}}$ be an *i.i.d.* random sequence defined on the probability space $(\Omega,\mathcal{F},P)$ and the state space $\mathcal{X} = \mathbb{R}^d$. A random sequence $S : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$ is called a **random walk** with step-size sequence X, if $S_0 \triangleq 0$ and $S_n \triangleq \sum_{i=1}^n X_i$ for $n \in \mathbb{N}$.

Remark 7. We can think of S_n as the random location of a particle after n steps, where the particle starts from origin and takes steps of size X_i at the ith step. From the *i.i.d.* nature of step-size sequence, we observe that $\mathbb{E}S_n = n\mathbb{E}X_1$ and $C_S(n,m) = (n \wedge m)\operatorname{Var}[X_1]$.

Remark 8. For the process $S : \Omega \to \mathcal{X}^{\mathbb{N}}$ it suffices to look at finite dimensional distributions for finite sets $[n] \subseteq \mathbb{N}$ for all $n \in \mathbb{N}$. If the *i.i.d.* step-size sequence X has a common density function, then from the transformation of random vectors, we can find the finite dimensional density

$$f_{S_1,\dots,S_n}(s_1,s_2,\dots,s_n) = \frac{1}{|J(s)|} f_{X_1,\dots,X_n}(s_1,s_2-s_1,\dots,s_n-s_{n-1}) = f_{X_1}(s_1)\prod_{i=2}^{n} f_{X_1}(s_i - s_{i-1}).$$

Recall that $J_{ij}(s) = \frac{\partial s_i}{\partial x_j} = \mathbb{1}_{\{j \leqslant i\}}$ and hence $J(s)$ is a lower triangular matrix with all non-zero entries being unity, and hence $|J(s)| = 1$.

Theorem 1.9. *The stochastic process $S : \Omega \to \mathbb{Z}_+^{\mathbb{Z}_+}$ has stationary and independent increments.*

Proof. One needs to show that for $(m_1,\dots,m_n) \subseteq \mathbb{Z}_+$, the joint distribution of increments $(S_{m_2} - S_{m_1},\dots,S_{m_n} - S_{m_{n-1}})$ is independent and distributed identically to $(S_{k+m_2} - S_{k+m_1},\dots,S_{k+m_n} - S_{k+m_{n-1}})$ for any and $k \in \mathbb{Z}_+$. For any $j \in [n-1]$, we define event space $\mathcal{E}_j \triangleq \sigma(X_{m_j+1},\dots,X_{m_{j+1}})$ and observe that the jth increment $S_{m_{j+1}} - S_{m_j}$ is measurable with respect to jth event space $\mathcal{E}_j$. Since X is an independent process, it follows that $(\mathcal{E}_1,\dots,\mathcal{E}_{n-1})$ are independent event spaces, and hence so are the increments. Stationarity of increments follow from the fact that the process X is *i.i.d.* and jth increment $S_{k+m_{j+1}} - S_{k+m_j}$ is sum of $m_{j+1} - m_j$ *i.i.d.* random variables, and hence an identical distribution for all $k \in \mathbb{Z}_+$. □

Corollary 1.10. *Let $p \in \mathbb{N}$ and for each $i \in [p]$ let $n \in \mathbb{N}^p, k \in \mathbb{Z}_+^p$ such that $n_1 \leqslant \dots \leqslant n_p$ and $k_1 \leqslant \dots \leqslant k_p$. Then, we can write the joint mass function*

$$P_{S_{n_1},\dots,S_{n_p}}(k_1,\dots,k_p) = P(\cap_{i\in[k]}\{S_{n_i} = k_i\}) = \prod_{i=1}^{p} P_{S_{n_i - n_{i-1}}}(k_i - k_{i-1}).$$

Proof. The result follows from stationary and independent increment property of the random walk S. □

Remark 9. For a one-dimensional random walk $S : \Omega \to \mathbb{Z}_+^{\mathbb{N}}$ with *i.i.d.* step size sequence $X : \Omega \to \{0,1\}^{\mathbb{N}}$ such that $P\{X_1 = 1\} = p$, the distribution for the random walk at nth step S_n is Binomial (n,p). That is,

$$P\{S_n = k\} = \binom{n}{k} p^k (1-p)^{n-k}, \quad k \in \{0,\dots,n\}.$$

1.4 Lévy processes

A right continuous with left limits stochastic process $X : \Omega \to \mathbb{R}^T$ for index set $T \subseteq \mathbb{R}_+$ with $X_0 = 0$ almost surely, is a **Lévy process** if the following conditions hold.

(L1) The increments are independent. For any instants $0 \leqslant t_1 < t_2 < \cdots < t_n < \infty$, the random variables $X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent.

(L2) The increments are stationary. For any instants $0 \leqslant t_1 < t_2 < \cdots < t_n < \infty$ and time-difference $s > 0$, the random vectors $(X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \dots, X_{t_n} - X_{t_{n-1}})$ and $(X_{s+t_2} - X_{s+t_1}, X_{s+t_3} - X_{s+t_2}, \dots, X_{s+t_n} - X_{s+t_{n-1}})$ are equal in distribution.

(L3) Continuous in probability. For any $\epsilon > 0$ and $t \geqslant 0$ it holds that $\lim_{h\to 0} P(\{|X_{t+h} - X_t| > \epsilon\}) = 0$.

Example 1.11. Two examples of Lévy processes are Poisson process and Wiener process. The distribution of Poisson process at time t is Poisson with rate λt and the distribution of Wiener process at time t is zero mean Gaussian with variance t.

Example 1.12. A random walk $S : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$ with *i.i.d.* step-size sequence $X : \Omega \to \mathcal{X}^{\mathbb{N}}$, is non-stationary with stationary and independent increments. To see non-stationarity, we observe that the mean $m_S(n) = n\mathbb{E}X_1$ depends on the step of the random walk. We have already seen the increment process of random walks.

1.5 Markov processes

Definition 1.13. A stochastic process X is **Markov** if conditioned on the present state, future is independent of the past. We denote the history of the process until time t as $\mathcal{F}_t = \sigma(X_s, s \leqslant t)$. That is, for any ordered index set T containing any two indices $u > t$, we have

$$P(\{X_u \leqslant x_u\} \mid \mathcal{F}_t) = P(\{X_u \leqslant x_u\} \mid \sigma(X_t)).$$

The range of the process is called the **state space**.

Remark 10. We next re-write the Markov property more explicitly for the process X. For all $x, y \in \mathcal{X}$, finite set $S \subseteq T$ such that $\max S < t < u$, and $H_S = \cap_{s \in S} \{X_s \leqslant x_s\} \in \mathcal{F}_t$, we have

$$P(\{X_u \leqslant y\} \mid H_S \cap \{X_t \leqslant x\}) = P(\{X_u \leqslant y\} \mid \{X_t \leqslant x\}).$$

Remark 11. When the state space $\mathcal{X}$ is countable, we can write $H_S = \cap_{s \in S} \{X_s = x_s\}$ and the Markov property can be written as

$$P(\{X_u = y\} \mid H_S \cap \{X_t = x\}) = P(\{X_u = x_u\} \mid \{X_t = x\}).$$

Remark 12. In addition, when the index set is countable, i.e. $T = \mathbb{Z}_+$, then we can take past as $S = \{0, \dots, n-1\}$, present as instant n, and the future as $n+1$. Then, the Markov property can be written as

$$P(\{X_{n+1} = y\} \mid H_{n-1} \cap \{X_n = x\}) = P(\{X_{n+1} = y\} \mid \{X_n = x\}),$$

for all $n \in \mathbb{Z}_+, x, y \in \mathcal{X}$.

We will study this process in detail in coming lectures.

Example 1.14. A random walk $S : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$ with *i.i.d.* step-size sequence $X : \Omega \to \mathcal{X}^{\mathbb{N}}$, is a homogeneous Markov sequence. For any $n \in \mathbb{Z}_+$ and $x, y, s_1, \dots, s_{n-1} \in \mathcal{X}$, we can write the conditional probability

$$P(\{S_{n+1} = y\} \mid \{S_n = x, S_{n-1} = s_{n-1}, \dots, S_1 = s_1\}) = P(\{S_{n+1} - S_n = y - x\}) = P(\{S_{n+1} = y\} \mid \{S_n = x\}).$$

Lemma 1.15. *The stochastic process $S : \Omega \to \mathbb{Z}_+^{\mathbb{Z}_+}$ is homogeneously Markov.*

Proof. Since the process has stationary and independent increments, we have

$$P(\{S_{n+m} = k\} \mid \{S_1 = k_1, S_2 = k_2, \dots, S_n = k_n\}) = P(\{S_{n+m} - S_n = k - k_n\}) = P(\{S_{n+m} = k\} \mid \{S_n = k_n\}).$$

$\square$