# Lecture-21: Recurrent and transient states

## 1 Recurrence and Transience

We will consider a random sequence $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$ with initial state $X_0 = x \in \mathcal{X}$ and the $k$th hitting times to state $y$ for all $k \in \mathbb{N}$ denoted by $\tau_k \triangleq \tau_X^{\{y\},k}$ inductively defined as $\tau_k \triangleq \inf\{n > \tau_{k-1} : X_n = y\}$ where $\tau_0 \triangleq 0$. We define the inter-return time sequence $H : \Omega \to \mathbb{N}^{\mathbb{N}}$ as $H_k \triangleq H_X^{\{y\},k} = \tau_k - \tau_{k-1}$ for all $k \in \mathbb{N}$.

**Definition 1.1.** For a random sequence $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$ with initial state $X_0 = x$,

(i) the **probability of hitting state** $y$ **eventually** is denoted by $f_{xy} \triangleq P_x\{\tau_1 < \infty\}$, and

(ii) the **probability of first visit to state** $y$ **at time** $n \in \mathbb{N}$ is denoted by $f_{xy}^{(n)} \triangleq P_x\{\tau_1 = n\}$.

*Remark* 1. We can write the finiteness of hitting time $\tau_1$ as the disjoint union $\{\tau_1 < \infty\} = \cup_{n\in\mathbb{N}}\{\tau_1 = n\}$. Therefore, $f_{xy} = \sum_{n\in\mathbb{N}} f_{xy}^{(n)}$.

*Remark* 2. If $f_{xy} = P_x\{\tau_1 < \infty\} = 1$ for all initial states $x \in \mathcal{X}$, then $\tau_1$ is almost surely finite and hence a stopping time.

**Definition 1.2.** From the initial state $x$, the distribution

(i) for the first hitting time to state $y$ is called the **first passage time distribution** and denoted by $((f_{xy}^{(n)} : n \in \mathbb{N}), 1 - f_{xy})$, and

(ii) for the first return time to state $x$ is called the **first recurrence time distribution** and denoted by $((f_{xx}^{(n)} : n \in \mathbb{N}), 1 - f_{xx})$.

**Definition 1.3.** A state $y \in \mathcal{X}$ is called **recurrent** if $f_{yy} = 1$, and is called **transient** if $f_{yy} < 1$.

**Definition 1.4.** For any state $y \in \mathcal{X}$, the **mean recurrence time** is denoted by $\mu_{yy} \triangleq \mathbb{E}_y\tau_1$.

*Remark* 3. The mean recurrence time for any transient state is infinite. For any recurrent state $y \in \mathcal{X}$, we write $\tau_1 = \tau_1 \mathbb{1}_{\{\tau_1<\infty\}} = \sum_{n\in\mathbb{N}} n\mathbb{1}_{\{\tau_1=n\}}$ almost surely, and the mean recurrence time is given by $\mu_{yy} = \sum_{n\in\mathbb{N}} n f_{yy}^{(n)}$.

**Definition 1.5.** For a recurrent state $y \in \mathcal{X}$,

(i) if the mean recurrence time is finite, then the state $y$ is called **positive recurrent**, and

(ii) if the mean recurrence time is infinite, then the state $y$ is called **null recurrent**.

**Proposition 1.6.** *For a homogeneous discrete Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$, we have*

$$P_x\{N_y(\infty) = k\} = \begin{cases} 1 - f_{xy}, & k = 0, \\ f_{xy}f_{yy}^{k-1}(1 - f_{yy}), & k \in \mathbb{N}. \end{cases}$$

*Proof.* We can write the event of zero visits to state $y$ as $\{N_y(\infty) = 0\} = \{\tau_1 = \infty\}$. Further, we can write the event of $m$ visits to state $y$ as

$$\{N_y(\infty) = k\} = \{\tau_k < \infty\} \cap \{\tau_{k+1} = \infty\} = \cap_{j=1}^k \{H_j < \infty\} \cap \{H_{k+1} = \infty\}, \quad k \in \mathbb{N}.$$

Recall that $H : \Omega \to \mathbb{N}^{\mathbb{N}}$ is an independent random sequence with subsequence $(H_k : k \geqslant 2)$ identically distributed, with $P_x\{H_j = n\} = P_y\{\tau_1 = n\}$ for all $j \geqslant 2$. Therefore, we get

$$P_x\{N_y(\infty) = k\} = P_x\{H_1 < \infty\}\prod_{j=2}^k P_x\{H_j < \infty\} P_x\{H_{k+1} = \infty\} = f_{xy}f_{yy}^{k-1}(1 - f_{yy}).$$

$\square$

**Corollary 1.7.** *For a homogeneous Markov chain $X$, we have $P_x\left\{N_y(\infty) < \infty\right\} = \mathbb{1}_{\left\{f_{yy}<1\right\}} + (1 - f_{xy})\mathbb{1}_{\left\{f_{yy}=1\right\}}$.*

*Proof.* We can write the event $\left\{N_y(\infty) < \infty\right\}$ as disjoint union of events $\left\{N_y(\infty) = k\right\}$, to get the result. $\square$

*Remark* 4. For a time homogeneous Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$, we have
(i) $P_x\left\{N_y(\infty) = \infty\right\} = f_{xy}\mathbb{1}_{\left\{f_{yy}=1\right\}}$, and
(ii) $P_y\left\{N_y(\infty) = \infty\right\} = \mathbb{1}_{\left\{f_{yy}=1\right\}}$.

**Corollary 1.8.** *The mean number of visits to state $y$, starting from a state $x$ is $\mathbb{E}_x N_y(\infty) = \frac{f_{xy}}{1-f_{yy}}\mathbb{1}_{\left\{f_{yy}<1\right\}} + \infty\mathbb{1}_{\left\{f_{xy}>0, f_{yy}=1\right\}}$.*

*Remark* 5. For any state $y \in \mathcal{X}$, we have $\mathbb{E}_y N_y(\infty) = \frac{f_{yy}}{1-f_{yy}}\mathbb{1}_{\left\{f_{yy}<1\right\}} + \infty\mathbb{1}_{\left\{f_{yy}=1\right\}}$. That is, the mean number of visits to initial state $y$ is finite iff the state $y$ is transient.

*Remark* 6. In particular, this corollary implies the following consequences.

i. A transient state is visited a finite amount of times almost surely. This follows from Corollary 1.7, since $P_x\left\{N_y(\infty) < \infty\right\} = 1$ for all transient states $y \in \mathcal{X}$ and any initial state $x \in \mathcal{X}$.

ii. A recurrent state is visited infinitely often almost surely. This also follows from Corollary 1.7, since $P_y\left\{N_y(\infty) < \infty\right\} = 0$ for all recurrent states $y \in \mathcal{X}$.

iii. In a finite state Markov chain, not all states may be transient.

*Proof.* To see this, we assume that for a finite state space $\mathcal{X}$, all states $y \in \mathcal{X}$ are transient. Then, we know that $N_y(\infty)$ is finite almost surely for all states $y \in \mathcal{X}$. It follows that, for any initial state $x \in \mathcal{X}$

$$0 \leqslant P_x\left\{\sum_{y\in\mathcal{X}} N_y(\infty) = \infty\right\} = P_x(\cup_{y\in\mathcal{X}}\left\{N_y(\infty) = \infty\right\}) \leqslant \sum_{y\in\mathcal{X}} P_x\left\{N_y(\infty) = \infty\right\} = 0.$$

It follows that $\sum_{y\in\mathcal{X}} N_y(\infty)$ is also finite almost surely for all states $y \in \mathcal{X}$ for finite state space $\mathcal{X}$. However, we know that $\sum_{y\in\mathcal{X}} N_y(\infty) = \sum_{k\in\mathbb{N}}\sum_{y\in\mathcal{X}} 1_{\{X_k=y\}} = \infty$. This leads to a contradiction. $\square$

**Proposition 1.9.** *For a homogeneous DTMC $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$, a state $y \in \mathcal{X}$ is recurrent iff $\sum_{k\in\mathbb{N}} p_{yy}^{(k)} = \infty$, and transient iff $\sum_{k\in\mathbb{N}} p_{yy}^{(k)} < \infty$.*

*Proof.* Recall that if the mean recurrence time to a state $y \in \mathcal{X}$ is $\mathbb{E}_y N_y(\infty) = \sum_{k\in\mathbb{N}} p_{yy}^{(k)}$ finite then the state is transient and infinite if the state is recurrent. $\square$

**Corollary 1.10.** *For a transient state $y \in \mathcal{X}$, the following limits hold $\lim_{n\to\infty} p_{xy}^{(n)} = 0$, and $\lim_{n\to\infty} \frac{\sum_{k=1}^{n} p_{xy}^{(k)}}{n} = 0$.*

*Proof.* For a transient state $y \in \mathcal{X}$ and any state $x \in \mathcal{X}$, we have $\mathbb{E}_x N_y(\infty) = \sum_{n\in\mathbb{N}} p_{xy}^{(n)} < \infty$. Since the series sum is finite, it implies that the limiting terms in the sequence $\lim_{n\to\infty} p_{xy}^{(n)} = 0$. Further, we can write $\sum_{k=1}^{n} p_{xy}^{(k)} \leqslant \mathbb{E}_x N_y(\infty) \leqslant M$ for some $M \in \mathbb{N}$ and hence $\lim_{n\to\infty} \frac{\sum_{k=1}^{n} p_{xy}^{(k)}}{n} = 0$. $\square$

**Lemma 1.11.** *For any state $y \in \mathcal{X}$, let $H : \Omega \to \mathbb{N}^{\mathbb{N}}$ be the sequence of almost surely finite inter-visit times to state $y$, and $N_y(n) = \sum_{k=1}^{n} 1_{\{X_k=y\}}$ be the number of visits to state $y$ in $n$ times. Then, $N_y(n) + 1$ is a finite mean stopping time with respect to the sequence $H$.*

*Proof.* We first observe that $N_y(n) + 1 \leqslant n + 1$ and hence has a finite mean for each $n \in \mathbb{N}$. Further, we observe that $\{N_y(n) + 1 = k\}$ can be completely determined by observing $H_1, \dots, H_k$, since

$$\{N_y(n) + 1 = k\} = \left\{ \sum_{j=1}^{k-1} H_j \leqslant n < \sum_{j=1}^{k} H_j \right\} \in \sigma(H_1, \dots, H_k).$$

$\square$

**Theorem 1.12.** *Let $x, y \in \mathcal{X}$ be such that $f_{xy} = 1$ and $y$ is recurrent. Then,* $\lim_{n\to\infty} \frac{\sum_{k=1}^{n} p_{xy}^{(k)}}{n} = \frac{1}{\mu_{yy}}$.

*Proof.* Let $y \in \mathcal{X}$ be recurrent. The proof consists of three parts. In the first two parts, we will show that starting from the state $y$, we have the limiting empirical average of mean number of visits to state $y$ is $\lim_{n\to\infty} \frac{1}{n} \mathbb{E}_y N_y(n) = \frac{1}{\mu_{yy}}$. In the third part, we will show that for any starting state $x \in \mathcal{X}$ such that $f_{xy} = 1$, we have the limiting empirical average of mean number of visits to state $y$ is $\lim_{n\to\infty} \frac{1}{n} \mathbb{E}_x N_y(n) = \frac{1}{\mu_{yy}}$.

Lower bound: We observe that $N_y(n) + 1$ is a stopping time with respect to inter-visit times $H$ from Lemma 1.11. Further, we have $\sum_{j=1}^{N_y(n)+1} H_j > n$. Applying Wald's Lemma to the random sum $\sum_{j=1}^{N_y(n)+1} H_j$ , we get $\mathbb{E}_y(N_y(n) + 1)\mu_{yy} > n$. Taking limits, we obtain $\liminf_{n\in\mathbb{N}} \frac{\sum_{k=1}^{n} p_{yy}^{(k)}}{n} \geqslant \frac{1}{\mu_{yy}}$.

Upper bound: Let $X_0 = y$ and consider a fixed positive integer $M \in \mathbb{N}$. Then $H$ is *i.i.d.* and we define truncated recurrence times $\bar{H} : \Omega \to [M]^{\mathbb{N}}$ for all $j \in \mathbb{N}$ as $\bar{H}_j \triangleq M \wedge H_j$. It follows that the sequence $\bar{H}$ is *i.i.d.* and $\bar{H}_j \leqslant H_j$ for all $j \in \mathbb{N}$. We define the mean of the truncated recurrence times as $\bar{\mu}_{yy} \triangleq \mathbb{E}_y \bar{H}_1$. From the monotonicity of truncation, we get $\bar{\mu}_{yy} \leqslant \mu_{yy}$.

We define the random variable $\bar{\tau}_k \triangleq \sum_{j=1}^{k} \bar{H}_j$ for all $k \in \mathbb{N}$, and $\bar{\tau}_k \leqslant \tau_k$ for all $k \in \mathbb{N}$. We can define the associated counting process that counts the number of truncated recurrences in first $n$ steps as $\bar{N}_y(n) \triangleq \sum_{k\in\mathbb{N}} \mathbb{1}_{\{\bar{\tau}_k \leqslant n\}}$ for all $n \in \mathbb{N}$. We conclude that $\bar{N}_y(n) + 1$ is a stopping time with respect to *i.i.d.* process $\bar{H}$, and $\bar{N}_y(n) \geqslant N_y(n)$ sample path wise. Further, we have

$$\sum_{j=1}^{\bar{N}_y(n)+1} \bar{H}_j = \bar{\tau}_{\bar{N}_y(n)+1} = \bar{\tau}_{\bar{N}_y(n)} + \bar{H}_{\bar{N}_y(n)+1} \leqslant n + M.$$

Applying Wald's Lemma to stopping time $N_y(n) + 1$ with respect to *i.i.d.* sequence $H$ and stopping time $\bar{N}_y(n) + 1$ with respect to *i.i.d.* sequence $\bar{H}$, and monotonicity of expectation, we get

$$\mathbb{E}_y(N_y(n) + 1)\bar{\mu}_{yy} \leqslant \mathbb{E}_y(\bar{N}_y(n) + 1)\bar{\mu}_{yy} \leqslant n + M.$$

Taking limits, we obtain $\limsup_{n\in\mathbb{N}} \frac{\sum_{k=1}^{n} p_{yy}^{(k)}}{n} \leqslant \frac{1}{\bar{\mu}_{yy}}$. Letting $M$ grow arbitrarily large, we obtain the upper bound.

Starting from $x$: Further, we observe that $p_{xy}^{(k)} = \sum_{s=0}^{k-1} f_{xy}^{(k-s)} p_{yy}^{(s)}$. Since $1 = f_{xy} = \sum_{k\in\mathbb{N}} f_{xy}^{(k)}$, we have

$$\sum_{k=1}^{n} p_{xy}^{(k)} = \sum_{k=1}^{n}\sum_{s=0}^{k-1} f_{xy}^{(k-s)} p_{yy}^{(s)} = \sum_{s=0}^{n-1} p_{yy}^{(s)} \sum_{k-s=1}^{n-s} f_{xy}^{(k-s)} = \sum_{s=0}^{n-1} p_{yy}^{(s)} - \sum_{s=0}^{n-1} p_{yy}^{(s)} \sum_{k>n-s} f_{xy}^{(k)}.$$

Since the series $\sum_{k\in\mathbb{N}} f_{xy}^{(k)}$ converges, we get $\lim_{n\to\infty} \frac{\sum_{k=1}^{n} p_{xy}^{(k)}}{n} = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} p_{yy}^{(k)}}{n}$.

$\square$