Lecture-24: Poisson Point Processes

1 Simple point processes

Consider the d-dimensional Euclidean space $\mathbb{R}^d$. The collection of Borel measurable subsets $\mathcal{B}(\mathbb{R}^d)$ of the above Euclidean space is generated by sets $B(x) \triangleq \left\{ y \in \mathbb{R}^d : y_i \leqslant x_i \right\}$ for $x \in \mathbb{R}^d$.

Definition 1.1. A **simple point process** is a random countable collection of *distinct points* $S : \Omega \to \mathcal{X}^{\mathbb{N}}$, such that the distance $\|S_n\| \to \infty$ as $n \to \infty$.

Remark 1. Since S is a simple point process, each point S_n is unique. Therefore, we can identify S as a random set of points in $\mathcal{X}$ and $S \cap A$ is the random set of points in A.

Remark 2. For any simple point process S, we have $P(\{S_n = S_m \text{ for any } n \neq m\}) = 0$ and $|S \cap A|$ is finite almost surely for any bounded set $A \in \mathcal{B}(\mathcal{X})$.

Example 1.2 (Simple point process on the half-line). We can simplify this definition for $d = 1$. When $\mathcal{X} = \mathbb{R}_+$, one can order the points of the process $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ to get ordered process $\tilde{S} : \Omega \to \mathbb{R}_+^{\mathbb{N}}$, such that $\tilde{S}_n = S_{(n)}$ is the nth order statistics of S. That is, $S_{(0)} \triangleq 0$, and $S_{(n)} \triangleq \inf\left\{ S_k > S_{(n-1)} : k \in \mathbb{N} \right\}$. such that $S_{(1)} < S_{(2)} < \cdots < S_{(n)} < \ldots$, and $\lim_{n \in \mathbb{N}} S_{(n)} = \infty$. We will call this an arrival process.

Definition 1.3. Corresponding to a point process $S : \Omega \to \mathcal{X}^{\mathbb{N}}$, we denote the number of points in a set $A \in \mathcal{B}(\mathcal{X})$ by

$$N(A) \triangleq |S \cap A| = \sum_{n \in \mathbb{N}} \mathbb{1}_A(S_n), \text{ where we have } N(\emptyset) = 0.$$

The resulting process $N : \Omega \to \mathbb{Z}_+{}^{\mathcal{B}(\mathcal{X})}$ is called a **counting process** for the point process $S : \Omega \to \mathcal{X}^{\mathbb{N}}$.

Remark 3. Let $A \in \mathcal{B}(\mathcal{X})^k$ be a bounded partition of $B \in \mathcal{B}(\mathcal{X})$. From the disjointness of $(A_1, \ldots, A_k)$, we have

$$N(B) = \sum_{n \in \mathbb{N}} \mathbb{1}_{\cup_{i=1}^k A_i}(S_n) = \sum_{n \in \mathbb{N}} \sum_{i=1}^k \mathbb{1}_{A_i}(S_n) = \sum_{i=1}^k \sum_{n \in \mathbb{N}} \mathbb{1}_{A_i}(S_n) = \sum_{i=1}^k N(A_i).$$

Definition 1.4. A counting process is **simple** if the underlying point process is simple.

Remark 4. For a simple counting process N, we have $N(\{x\}) \leqslant 1$ almost surely for all $x \in \mathcal{X}$.

Remark 5. Let $N : \Omega \to \mathbb{Z}_+{}^{\mathcal{B}(\mathcal{X})}$ be the counting process for the point process $S : \Omega \to \mathcal{X}^{\mathbb{N}}$.

i_ Note that the point process S and the counting process N carry the same information.

ii_ The distribution of point process S is completely characterized by the finite dimensional distributions of random vectors $(N(A_1), \ldots, N(A_k))$ for any bounded sets $A_1, \ldots, A_k \in \mathcal{B}(\mathcal{X})$ and finite $k \in \mathbb{N}$.

Example 1.5 (Simple point process on the half-line). Since the Borel measurable sets $\mathcal{B}(\mathbb{R}_+)$ are generated by half-open intervals $\{(0, t] : t \in \mathbb{R}_+\}$, we denote the counting process by $N : \Omega \to \mathbb{Z}_+{}^{\mathbb{R}_+}$, where $N_t \triangleq N(0, t] = \sum_{n \in \mathbb{N}} \mathbb{1}_{\{S_n \in (0,t]\}}$ is the number of points in the half-open interval $(0, t]$. For $s < t$, the number of points in interval $(s, t]$ is $N(s, t] = N(0, t] - N(0, s] = N_t - N_s$.

Theorem 1.6 (Rényi). *Distribution of a simple point process $S : \Omega \to \mathcal{X}^{\mathbb{N}}$ on a locally compact second countable space $\mathcal{X}$ is completely determined by void probabilities $(P\{N(A) = 0\} : A \in \mathcal{B}(\mathcal{X}))$.*

Proof. It suffices to show that the finite dimensional distributions of S on locally compact sets are characterized by void probabilities.

Step 1: We will show this by induction on the number of points k in a bounded set $A \in \mathcal{B}$. Let $A_1, \dots, A_k, B \in \mathcal{B}(\mathcal{X})$ locally compact, then we will show that $u_k \triangleq P(\cap_{i=1}^k \{N(A_i) > 0\} \cap \{N(B) = 0\})$ can be computed from void probabilities. From $k = 1$, we have

$$P\{N(A_1) > 0, N(B) = 0\} = P\{N(B) = 0\} - P\{N(B \cup A_1) = 0\}.$$

The induction can be proved by the recursive relation

$$u_k = P(\cap_{i=1}^{k-1} \{N(A_i) > 0\} \cap \{N(B) = 0\}) - P(\cap_{i=1}^{k-1} \{N(A_i) > 0\} \cap \{N(A_k \cup B) = 0\}).$$

Step 2: For any locally compact set $B \in \mathcal{B}(\mathcal{X})$, there exists a sequence of nested partitions $B_n \triangleq (B_{n,j} : j \in [J_n])$ that eventually separates the points in $S \cap B$ as $n \to \infty$. We define the number of subsets of partition $(B_{n,j} : j \in [J_n])$ that consist of at least one point in $S \cap B$, as $H_n(B) \triangleq \sum_{j=1}^{J_n} \mathbb{1}_{\{N(B_{n,j}) > 0\}}$ where $H_n(B) \uparrow N(B)$ almost surely.

Step 3: We next show that for all locally compact sets $B_1, \dots, B_k \in \mathcal{B}(\mathcal{X})$ and $j_1, \dots, j_k \in \mathbb{N}$, the probability $P(\cap_{i=1}^k \{H_n(B_i) = j_i\})$ can be expressed in terms of void probabilities. We observe that

$$P(\cap_{i=1}^k \{H_n(B) = j_i\}) = \sum_{T_1, \dots, T_k \subseteq [J_n] : |T_1| = j_1, \dots, |T_k| = j_k} P\Big(\cap_{i=1}^k \cap_{j \in T_i} \Big\{N(B_{n,j}^i) > 0\Big\} \cap \Big\{N(\cup_{j \notin \cup_{i=1}^k T_i} B_{n,j}^i) = 0\Big\}\Big).$$

This can be expressed in terms of void probabilities by Step 1.

Step 4: For a simple point process, we have the following almost sure limit $\lim_n \cap_{i=1}^k \{H_n(B_i) = j_i\} = \cap_{i=1}^k \{N(B_i) = j_i\}$. The result follows from the continuity of probability.

□

Remark 6. Recall that $|A| = \int_{x \in A} dx$ is the volume of the set $A \in \mathcal{B}(\mathbb{R}^d)$ and for any such A.

Definition 1.7. The **intensity measure** $\Lambda : \mathcal{B}(\mathcal{X}) \to \mathbb{R}_+$ is defined for each bounded set $A \in \mathcal{X}$ as its scaled volume in terms of the **intensity density** $\lambda : \mathbb{R}^d \to \mathbb{R}_+$, as

$$\Lambda(A) \triangleq \int_{x \in A} \lambda(x) dx.$$

If the intensity density $\lambda(x) = \lambda$ for all $x \in \mathbb{R}^d$, then $\Lambda(A) = \lambda |A|$. In particular for partition $A_1, \dots, A_k$ for a set B, we have $\Lambda(B) = \sum_{i=1}^k \Lambda(A_i)$.

2 Poisson point process

Definition 2.1. A non-negative integer valued random variable $N : \Omega \to \mathbb{Z}_+$ is called **Poisson** if for some constant $\lambda > 0$, we have

$$P\{N = n\} = e^{-\lambda} \frac{\lambda^n}{n!}.$$

Remark 7. It is easy to check that $\mathbb{E}N = \mathrm{Var}[N] = \lambda$. Furthermore, the moment generating function $M_{N_t} = \mathbb{E}e^{tN} = e^{\lambda(e^t - 1)}$ exists for all $t \in \mathbb{R}$.

Corollary 2.2. *A simple counting process $N : \Omega \to \mathbb{Z}_+^{\mathcal{B}(\mathcal{X})}$ has Poisson marginal distribution with intensity measure $\Lambda : \mathcal{B}(\mathcal{X}) \to \mathbb{R}_+$ if and only if void probabilities are exponential with the same intensity measure Λ.*

Proof. It is clear that if the marginal distribution of the counting process N is Poisson with intensity measure Λ, then the void probability $P\{N(A)=0\} = e^{-\Lambda(A)}$ is exponential for any bounded set $A \in \mathcal{B}(\mathcal{X})$.

Conversely, we assume that the void probabilities are exponentially distributed with intensity measure Λ. It follows from the linearity of intensity measure that for any finite, bounded, and disjoint sets $B_1,\dots,B_k \in \mathcal{B}(\mathcal{X})$, we have

$$P(\cap_{i=1}^k \{N(B_i)=0\}) = P\left\{N(\cup_{i=1}^k B_i) = 0\right\} = e^{-\Lambda(\cup_{i=1}^k B_i)} = \prod_{i=1}^k e^{-\Lambda(B_i)} = \prod_{i=1}^k P\{N(B_i)=0\}.$$

That is, the Bernoulli random vector $(\mathbb{1}_{\{N(B_i)=0\}} : i \in [k])$ is independent for any finite $k \in \mathbb{N}$ and bounded disjoint $\mathcal{B}(\mathcal{X})$ measurable sets $B_1,\dots,B_k$. Next we consider a set $B \in \mathcal{B}(\mathcal{X})$ and a partition $B_n \triangleq (B_{n,j} : j \in [J_n])$ of B such that $\Lambda(B_{n,j}) = \frac{\Lambda(B)}{J_n}$ for all $j \in [J_n]$. It follows that $H_n(B) \triangleq \sum_{j=1}^{J_n} \mathbb{1}_{\{N(B_{n,j})>0\}}$ is the sum of J_n *i.i.d.* Bernoulli random variables with success probability $p_n \triangleq 1 - e^{-\Lambda(B)/J_n}$, and hence has a Binomial distribution with parameters (J_n, p_n). Therefore,

$$P\{H_n(B)=m\} = \frac{e^{-\Lambda(B)}}{m!}\binom{J_n}{m}(e^{\Lambda(B)/J_n}p_n)^m = e^{-\Lambda(B)}\frac{J_n!}{(J_n-m)!}\left(e^{\Lambda(B)/J_n}-1\right)^m.$$

Recall that $H_n(B) \uparrow N(B)$ as $n \to \infty$ in the proof of Rényi's Theorem, and $\lim_{n\to\infty} J_n = \infty$ and $\lim_{n\in\mathbb{N}} |B_{n,j}| = 0$. Thus, $\lim_{n\to\infty} \frac{J_n!}{(J_n-m)!}(e^{\Lambda(B)/J_n}-1)^m = \Lambda(B)^m$. Taking limit $n \to \infty$ on both sides of the above equation, we get the result. □

Definition 2.3. A counting process $N : \Omega \to \mathbb{Z}_+^{\mathcal{B}(\mathcal{X})}$ has the **completely independence property**, if for any collection of finite disjoint and bounded sets $A_1,\dots,A_k \in \mathcal{B}(\mathcal{X})$, the vector $(N(A_1),\dots,N(A_k)) : \Omega \to \mathbb{Z}_+^k$ is independent. That is,

$$P\left(\bigcap_{i=1}^k \{N(A_i)=n_i\}\right) = \prod_{i=1}^k P\{N(A_i)=n_i\}, \quad n \in \mathbb{Z}_+^k.$$

Definition 2.4. A simple point process $S : \Omega \to \mathcal{X}^{\mathbb{N}}$ is **Poisson point process**, if the associated counting process $N : \Omega \to \mathbb{Z}_+^{\mathcal{B}(\mathcal{X})}$ has complete independence property and the marginal distributions are Poisson.

Definition 2.5. The **intensity measure** $\Lambda : \mathcal{B}(\mathcal{X}) \to \mathbb{R}_+$ of Poisson process S is defined by $\Lambda(A) \triangleq \mathbb{E}N(A)$ for all bounded $A \in \mathcal{B}(\mathcal{X})$.

Remark 8. Recall that for any partition $A \in \mathcal{B}(\mathcal{X})^k$ of a bounded set $B \in \mathcal{B}(\mathcal{X})$, we have $N(B) = \sum_{i=1}^k N(A_i)$ and therefore it follows from the linearity of expectations that $\Lambda(B) = \mathbb{E}N(B) = \sum_{i=1}^k \mathbb{E}N(A_i) = \sum_{i=1}^k \Lambda(A_i)$. Thus, this is a valid intensity measure.

Remark 9. For a Poisson process with intensity measure Λ, it follows from the definition that for any finite $k \in \mathbb{Z}_+$, and bounded mutually disjoint sets $A_1,\dots,A_k \in \mathcal{B}(\mathcal{X})$, we have

$$P\left(\cap_{i=1}^k \{N(A_i)=n_i\}\right) = \prod_{i=1}^k \left(e^{-\Lambda(A_i)}\frac{\Lambda(A_i)^{n_i}}{n_i!}\right), \quad n \in \mathbb{Z}_+^k.$$

Definition 2.6. If the intensity measure Λ of a Poisson process S satisfies $\Lambda(A) = \lambda|A|$ for all bounded $A \in \mathcal{B}(\mathcal{X})$, then we call S a **homogeneous Poisson point process** and λ is its intensity.

3 Equivalent characterizations

Theorem 3.1 (Equivalences). *Following are equivalent for a simple counting process* $N : \Omega \to {\mathbb{Z}_+}^{\mathcal{B}(\mathcal{X})}$.

i_ Process N is Poisson with locally finite intensity measure Λ.

ii$_-$ *For each bounded $A \in \mathcal{B}(\mathcal{X})$, we have $P\{N(A) = 0\} = e^{-\Lambda(A)}$.*

iii$_-$ *For each bounded $A \in \mathcal{B}(\mathcal{X})$, the number of points $N(A)$ is a Poisson with parameter $\Lambda(A)$.*

iv$_-$ *Process N has the completely independence property, and $\mathbb{E}N(A) = \Lambda(A)$ for all bounded sets $A \in \mathcal{B}(\mathcal{X})$.*

Proof. We will show that $i_- \implies ii_- \implies iii_- \implies iv_- \implies i_-$.

$i \implies ii_-$ It follows from the definition of Poisson point processes and definition of Poisson random variables.

$ii \implies iii_-$ From Corollary 2.2, we know that if void probabilities are exponential, then the marginal distributions are Poisson.

$iii \implies iv_-$ We will show this in two steps.

Mean: Since the distribution of random variable $N(A)$ is Poisson, it has mean $\mathbb{E}N(A) = \Lambda(A)$.

CIP: Consider a partition $A \in \mathcal{B}^k$ for a bounded set $B \in \mathcal{B}(\mathcal{X})$, then $\Lambda(B) = \Lambda(A_1) + \cdots + \Lambda(A_k)$. Consider all partitions $n \in \mathbb{Z}_+^k$ of a non-negative integer $m \in \mathbb{Z}_+$, to write

$$P\{N(B) = m\} = \sum_{n_1+\cdots+n_k=m} P\{N(A_1) = n_1, \ldots, N(A_k) = n_k\}.$$

Using the definition of Poisson distribution, we can write the LHS of the above equation as

$$P\{N(B) = m\} = e^{-\Lambda(B)}\frac{\Lambda(B)^m}{m!} = \prod_{i=1}^k e^{-\Lambda(A_i)}\frac{(\sum_{i=1}^k \Lambda(A_i))^m}{m!}.$$

Since the expansion of $(a_1 + \cdots + a_k)^m = \sum_{n_1+\cdots+n_k=m}\binom{m}{n_1,\ldots,n_k}\prod_{i=1}^k a_i^{n_i}$, we get

$$P\{N(B) = m\} = \frac{1}{m!}\sum_{n_1+\cdots+n_k=n}\binom{m}{n_1,\ldots,n_k}\prod_{i=1}^k e^{-\Lambda(A_i)}\Lambda(A_i))^{n_i} = \sum_{n_1+\cdots+n_k=m}\left(\prod_{i=1}^k e^{-\Lambda(A_i)}\frac{\Lambda(A_i))^{n_i}}{n_i!}\right).$$

Equating each term in the summation, we get $P\{N(A_1) = n_1, \ldots, N(A_k) = n_k\} = \prod_{i=1}^k P\{N(A_i) = n_i\}$.

$iv \implies i_-$ From Corollary 2.2, if the void probability is exponential with intensity measure Λ, then the marginal distribution if Poisson with the same intensity measure. We define $f : \mathcal{B}(\mathcal{X}) \to (-\infty, 0]$ by $f(A) \triangleq \ln P\{N(A) = 0\}$ for all bounded $A \in \mathcal{B}(\mathcal{X})$. Then, we observe that for any partition $(A_1, \ldots, A_k)$ of A, we have $f(\cup_{i=1}^k A_i) = \ln P\{N(A) = 0\} = \ln\prod_{i=1}^k P\{N(A_i) = 0\} = \sum_{i=1}^k f(A_i)$. It follows that $-f : \mathcal{B}(\mathcal{X}) \to \mathbb{R}_+$ is an intensity measure, and $P\{N(A) = 0\} = e^{f(A)}$. Since $\mathbb{E}N(A) = -f(A) = \Lambda(A)$, the result follows.

□

Corollary 3.2 (Poisson process on the half-line). *A random process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ indexed by time $t \in \mathbb{Z}_+$ is the counting process associated with a one-dimensional Poisson process $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ having intensity measure Λ iff*

(a) Starting with $N_0 = 0$, the process N_t takes a non-negative integer value for all $t \in \mathbb{R}_+$;

(b) the increment $N_s - N_t$ is surely nonnegative for any $s \geqslant t$;

(c) the increments $N_{t_1}, N_{t_2} - N_{t_1}, \ldots, N_{t_n} - N_{t_{n-1}}$ are independent for any $0 < t_1 < t_2 < \cdots < t_{n-1} < t_n$;

(d) the increment $N_s - N_t$ is distributed as Poisson random variable with parameter $\Lambda(t, s]$ for $s \geqslant t$.

The Poisson process is homogeneous with intensity λ, iff in addition to conditions $(a), (b), (c)$, the distribution of the increment $N_{t+s} - N_t$ depends on the value $s \in \mathbb{R}_+$ but is independent of $t \in \mathbb{R}_+$. That, is the increments are stationary.

Proof. We have already seen that definition of Poisson processes implies all four conditions. Conditions (a) and (b) imply that N is a simple counting process on the half-line, condition (c) is the complete independence property of the point process, and condition (d) provides the intensity measure. The result follows from the equivalence iv_- in Theorem 3.1. □