Lecture-25: Poisson processes: Conditional distribution

1 Joint conditional distribution of points in a finite window

Let $\mathcal{X} = \mathbb{R}^d$ be a d-dimensional Euclidean space, and $S : \Omega \to \mathcal{X}^{\mathbb{N}}$ be a Poisson point process with intensity measure $\Lambda : \mathcal{B}(\mathcal{X}) \to \mathbb{R}_+$ and associated counting process $N : \Omega \to \mathbb{Z}_+^{\mathcal{B}(\mathcal{X})}$.

Proposition 1.1. *Let $k \in \mathbb{N}$ be any positive integer. Consider a Poisson point process $S : \Omega \to \mathcal{X}^{\mathbb{N}}$ with intensity measure $\Lambda : \mathcal{B}(\mathcal{X}) \to \mathbb{R}_+$, a finite partition $A \in \mathcal{B}(\mathcal{X})^k$ that partitions a bounded set $B \in \mathcal{B}(\mathcal{X})$, and a vector $n \in \mathbb{Z}_+^k$ that partitions a non negative integer $m \in \mathbb{Z}_+$. Then,*

$$P(\{N(A_1) = n_1, \dots, N(A_k) = n_k\} \mid \{N(B) = m\}) = \binom{m}{n_1, \dots, n_k} \prod_{i=1}^{k} \left(\frac{\Lambda(A_i)}{\Lambda(B)}\right)^{n_i}. \tag{1}$$

Proof. From the definition of conditional probability and the fact that $\cap_{i=1}^{k} \{N(A_i) = n_i\} \subseteq \{N(B) = m\}$, we can write the conditional probability on LHS as the ratio

$$\frac{P\{N(A_1) = n_1, \dots, N(A_k) = n_k, N(B) = m\}}{P\{N(B) = m\}} = \frac{P\{N(A_1) = n_1, \dots, N(A_k) = n_k\}}{P\{N(B) = m\}}.$$

From the complete independence property and Poisson marginals for the joint distribution of $(N(A_1), \dots, N(A_k))$ for the partition $A \in \mathcal{B}(\mathcal{X})^k$, and the fact that the intensity measure sums over disjoint sets, i.e. $\Lambda(B) = \sum_{i=1}^{k} \Lambda(A_i)$, we can rewrite the RHS of the above equation as

$$\frac{P\{N(A_1) = n_1, \dots, N(A_k) = n_k\}}{P\{N(B) = m\}} = \frac{\prod_{i=1}^{k} e^{-\Lambda(A_i)} \frac{\Lambda(A_i)^{n_i}}{n_i!}}{e^{-\Lambda(B)} \frac{\Lambda(B)^m}{m!}} = \binom{m}{n_1, \dots, n_k} \prod_{i=1}^{k} \left(\frac{\Lambda(A_i)}{\Lambda(B)}\right)^{n_i}.$$

□

Remark 1. Consider a Poisson point process $S : \Omega \to \mathcal{X}^{\mathbb{N}}$ with intensity measure $\Lambda : \mathcal{B}(\mathcal{X}) \to \mathbb{R}_+$ and counting process $N : \Omega \to \mathbb{Z}_+^{\mathcal{B}(\mathcal{X})}$. Let $A \in \mathcal{B}(\mathcal{X})^k$ be a partition for bounded set $B \in \mathcal{B}(\mathcal{X})$.

i_ Defining $p_i \triangleq \frac{\Lambda(A_i)}{\Lambda(B)}$, we see that $(p_1, \dots, p_k) \in \mathcal{M}([k])$ is a probability distribution. We also observe that

$$p_i = P(\{N(A_i) = 1\} \mid \{N(B) = 1\}) = P(\{|S \cap A_i| = 1\} \mid \{|S \cap B| = 1\}).$$

When $N(B) = 1$, we can call the point of S in B as S_1 without any loss of generality. That is, if we call $\{S_1\} = S \cap B$, then we have

$$p_i = P(\{S_1 \in A_i\} \mid \{S_1 \in B\}).$$

Similarly, when $N(B) = n_i$, we call the points of S in B as $S_1, \dots, S_{n_i}$ and denote $S \cap B = \{S_1, \dots, S_{n_i}\}$. For this case, we observe

$$\begin{aligned} P(\{N(A_i) = n_i\} \mid \{N(B) = n_i\}) &= P(\{|S \cap A_i| = n_i\} \mid \{|S \cap B| = n_i\}) = p_i^{n_i} \\ &= P(\cap_{j=1}^{n_i} \{S_j \in A_i\} \mid \{\{S_1, \dots, S_{n_i}\} = S \cap B\}) = \prod_{j=1}^{n_i} P(\{S_j \in A_i\} \mid \{S_j \in B\}). \end{aligned}$$

ii. We can rewrite the Equation (1) as a multinomial distribution, where

$$P(\{N(A_1) = n_1, \dots, N(A_k) = n_k\} \mid \{N(B) = m\}) = \binom{m}{n_1, \dots, n_k} p_1^{n_1} \dots p_k^{n_k}.$$

iii. For any finite set $F \subseteq \mathbb{N}$ of size $m \in \mathbb{Z}_+$ and $n \in \mathbb{Z}_+^k$ a partition of m, we define $\mathcal{P}_k(F, n)$ to be the collection of all k-partitions $E \in \mathcal{P}(\mathbb{N})^k$ of F such that $|E_i| = n_i$ for $i \in [k]$. Then, the multinomial coefficient accounts for number of partitions of m points into sets with $n_1, \dots, n_k$ points. That is,

$$\binom{m}{n_1, \dots, n_k} = |\mathcal{P}_k([m], n)|.$$

iv. Recall that the event $\{N(A_i) = n_i\} = \{|S \cap A_i| = n_i\}$. Hence, we can write

$$\begin{aligned} P(\cap_{i=1}^k \{|S \cap A_i| = n_i\} \mid \{|S \cap B| = m\}) &= \binom{m}{n_1, \dots, n_k} p_1^{n_1} \dots p_k^{n_k} \\ &= \sum_{E \in \mathcal{P}_k(S \cap B, n)} \prod_{i=1}^k \prod_{S_j \in E_i} P(\{S_j \in A_i\} \mid \{S_j \in B\}). \end{aligned}$$

v. When $N(B) = m$, we denote $S \cap B$ by $F = \{S_1, \dots, S_m\}$ without any loss of generality. We further observe that when $N(A_i) = n_i$ for all $i \in [k]$, then $(S \cap A_1, \dots, S \cap A_k) \in \mathcal{P}_k(S \cap B, n)$. Therefore, we can re-write the event

$$\cap_{i=1}^k \{N(A_i) = n_i\} = \cap_{i=1}^k \{|S \cap A_i| = n_i\} = \cup_{E \in \mathcal{P}_k(S \cap B, n)} (\cap_{i=1}^k \{S \cap A_i = E_i\}).$$

That is, we can write the conditional probability conditioned on $S \cap B = F$, as

$$\begin{aligned} P(\cap_{i=1}^k \{N(A_i) = n_i\} \mid \{N(B) = m\}) &= \sum_{E \in \mathcal{P}_k(F, n)} P(\cap_{i=1}^k \{S \cap A_i = E_i\} \mid \{S \cap B = F\}) \\ &= \sum_{E \in \mathcal{P}_k(F, n_1, \dots, n_k)} P(\cap_{i=1}^k \cap_{S_j \in E_i} \{S_j \in A_i\} \mid \{S \cap B = F\}). \end{aligned}$$

vi. Equating the RHS of the above equation term-wise, we obtain that conditioned on each of these points falling inside the window B, the conditional probability of each point falling in partition A_i is independent of all other points and given by p_i. That is, we have

$$P(\cap_{i=1}^k \cap_{S_j \in E_i} \{S_j \in A_i\} \mid \{S \cap B = F\}) = \prod_{i=1}^k \prod_{S_j \in E_i} P(\{S_j \in A_i\} \mid \{S_j \in B\}) = \prod_{i=1}^k p_i^{n_i} = \prod_{i=1}^k \left(\frac{\Lambda(A_i)}{\Lambda(B)}\right)^{n_i}.$$

It means that given m points in the window B, the location of these points are independently and identically distributed in B according to the distribution $\frac{\Lambda(\cdot)}{\Lambda(B)}$.

vii. If the Poisson process is homogeneous, the distribution is uniform over the window B.

viii. For a Poisson process with intensity measure Λ and any bounded set $A \in \mathcal{B}$, the number of points $N(A)$ in the set A is a Poisson random variable with parameter $\Lambda(A)$. Given the number of points $N(A)$, the location of all the points in $S \cap A$ are *i.i.d.* with density $\frac{\lambda(x)}{\Lambda(A)}$ for all $x \in A$.

Remark 2 (Simulating a homogeneous Poisson point process). If we are interested in simulating a two dimensional homogeneous Poisson point process with density λ in a uniform square $A = [0, 1] \times [0, 1]$. Then, we first generate the random variable $N(A) : \Omega \to \mathbb{Z}_+$ that takes value n with probability $e^{-\lambda} \frac{\lambda^n}{n!}$. Next, for each of the $N(A) = n$ points, we generate the location $(X_i, Y_i) \in \mathbb{R}^2$ uniformly at random. That is, $X : \Omega \to [0, 1]^n$ and $Y : \Omega \to [0, 1]^n$ are independent *i.i.d.* uniform sequences.

Corollary 1.2. *For a homogeneous Poisson point process on the half-line with ordered set of points $\tilde{S} = (S_{(n)} \in \mathbb{R}_+ : n \in \mathbb{N})$, we can write the conditional density of ordered points $(S_{(1)}, \dots, S_{(k)})$ given the event $\{N_t = k\}$ as the ordered statistics of* i.i.d. *uniformly distributed random variables. Specifically, we have*

$$f_{S_{(1)},\dots,S_{(k)} \mid N_t=k}(t_1, \dots, t_k) = k! \frac{\mathbb{1}_{\{0<t_1 \leqslant \dots \leqslant t_k \leqslant t\}}}{t^k}.$$

Proof. Given $\{N_t = k\}$, we can denote the points of the Poisson process in $(0, t]$ by $S_1, \dots, S_k$. From the above remark, we know that $S_1, \dots, S_k$ are *i.i.d.* uniform in $(0, t]$, conditioned on the number of points $N_t = k$. Hence, we can write

$$F_{S_1,\dots,S_k \mid N_t=k}(t_1, \dots, t_k) = P(\cap_{i=1}^k \{S_i \in (0, t_i]\} \mid \{N_t = k\}) = \prod_{i=1}^k P(\{S_i \in (0, t_i]\} \mid \{S_i \in (0, t]\}) = \prod_{i=1}^k \frac{t_i}{t} \mathbb{1}_{\{0<t_i \leqslant t\}}.$$

Therefore, for $0 < t_1 < \dots < t_k < 1$ and $(h_1, \dots, h_k)$ sufficiently small, we have

$$P\Big(\cap_{i=1}^k \{S_i \in (t_i, t_i + h_i]\} \Big) = \prod_{i=1}^k \frac{h_i}{t}.$$

Since $(S_1, \dots, S_k)$ are conditionally independent given $S \cap A = \{S_1, \dots, S_k\}$, it follows that any permutation $\sigma : [k] \to [k]$, the conditional joint distribution of $(S_{\sigma(1)}, \dots, S_{\sigma(k)})$ is identical to that of $(S_1, \dots, S_k)$ given $S \cap A = \{S_1, \dots, S_k\}$. Further, we observe that the order statistics of $(S_{\sigma(1)}, \dots, S_{\sigma(k)})$ is identical to that of $(S_1, \dots, S_k)$. Therefore, we can write the following equality for the events

$$\cap_{i=1}^k \Big\{ S_{(i)} \in (t_i, t_i + h_i] \Big\} = \cup_{\sigma:[k]\to[k] \text{ permutation}} \cap_{i=1}^k \Big\{ S_{\sigma(i)} \in (t_i, t_i + h_i] \Big\}.$$

The result follows since the number of permutations $\sigma : [k] \to [k]$ is $k!$. $\square$