Lecture-27: Poisson process on the half-line

1 Simple point processes on the half-line

A stochastic process defined on the half-line $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ is a **counting process** if

1. $N_0 = 0$, and
2. for each $\omega \in \Omega$, the sample path $N(\omega) : \mathbb{R}_+ \to \mathbb{Z}_+$ is non-decreasing, integer valued, and right continuous function of time $t \in \mathbb{R}_+$.

Each discontinuity of the sample path of the counting process can be thought of as a jump of the process, as shown in Figure 1. A simple counting process has the unit jump size almost surely. General point processes in higher dimension don't have any inter-arrival time interpretation.

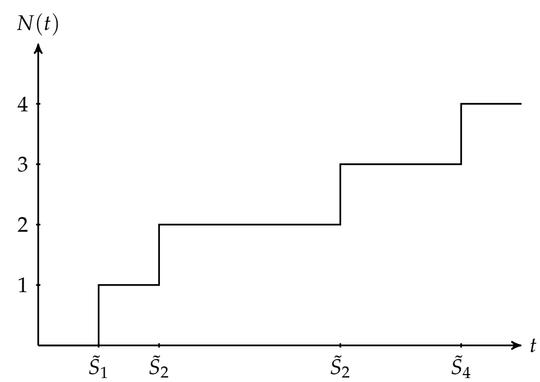

Figure 1: Sample path of a simple counting process.

Definition 1.1. The points of discontinuity are also called the **arrival instants** of the counting process N. The n**th arrival instant** is a random variable denoted $\tilde{S}_n : \Omega \to \mathbb{R}_+$, defined inductively as

$$\tilde{S}_0 \triangleq 0, \qquad \tilde{S}_n \triangleq \inf\{t \geqslant 0 : N_t \geqslant n\}, \quad n \in \mathbb{N}.$$

Definition 1.2. The **inter arrival time** between $(n-1)$th and nth arrival is denoted by X_n and written as $X_n \triangleq \tilde{S}_n - \tilde{S}_{n-1}$.

Remark 1. For a simple point process, we have $P\{X_n = 0\} = P\{X_n \leqslant 0\} = 0$.

Lemma 1.3. *Simple counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ and arrival process $\tilde{S} : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ are inverse processes, i.e.*

$$\left\{\tilde{S}_n \leqslant t\right\} = \{N_t \geqslant n\}.$$

Proof. Let $\omega \in \left\{\tilde{S}_n \leqslant t\right\}$, then $N_{\tilde{S}_n} = n$ by definition. Since N is a non-decreasing process, we have $N_t \geqslant N_{\tilde{S}_n} = n$. Conversely, let $\omega \in \{N_t \geqslant n\}$, then it follows from definition that $\tilde{S}_n \leqslant t$. □

Corollary 1.4. *For arrival instants $\tilde{S} : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ associated with a counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ we have $\left\{\tilde{S}_n \leqslant t, \tilde{S}_{n+1} > t\right\} = \{N_t = n\}$ for all $n \in \mathbb{Z}_+$ and $t \in \mathbb{R}_+$.*

Proof. It is easy to see that $\{\tilde{S}_{n+1} > t\} = \{\tilde{S}_{n+1} \leqslant t\}^c = \{N_t \geqslant n+1\}^c = \{N_t < n+1\}$. Hence,

$$\{N_t = n\} = \{N_t \geqslant n, N_t < n+1\} = \{\tilde{S}_n \leqslant t, \tilde{S}_{n+1} > t\}.$$

□

Lemma 1.5. *Let $F_n(x)$ be the distribution function for S_n, then $P_n(t) \triangleq P\{N_t = n\} = F_n(t) - F_{n+1}(t)$.*

Proof. It suffices to observe that following is a union of disjoint events,

$$\{\tilde{S}_n \leqslant t\} = \{\tilde{S}_n \leqslant t, \tilde{S}_{n+1} > t\} \cup \{\tilde{S}_n \leqslant t, \tilde{S}_{n+1} \leqslant t\}.$$

□

2 IID exponential inter-arrival times characterization

Proposition 2.1. *The counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ associated with a simple Poisson point process $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is Markov.*

Proof. We define the event space $\mathcal{F}_t \triangleq \sigma(N_s : s \leqslant t)$ as the history of the process until time $t \in \mathbb{R}_+$. Then, from the independent increment property of Poisson processes, we have for any historical event $H_s \in \mathcal{F}_s$

$$P(\{N_t = n\} \mid H_s \cap \{N_s = k\}) = P(\{N_t - N_s = n-k\} \mid H_s \cap \{N_s = k\}) = P(\{N_t = n\} \mid \{N_s = k\}).$$

The transition probability matrix is $P(s,t)$ with its (k,n)th entry given by $e^{-\Lambda(s,t]} \frac{(\Lambda(s,t])^{n-k}}{(n-k)!}$. □

Remark 2. A Markov process $X : \Omega \to \mathcal{X}^{\mathbb{R}}$ is time homogeneous if the transition matrix $P(s,t) = P(t-s)$ for all $t \geqslant s$. Thus the counting process for a homogeneous Poisson point process is time homogeneous Markov process, as the transition probability matrix $P(s,t) = P(t-s)$ with its (k,n)th entry given by $e^{-\lambda(t-s)} \frac{(\lambda(t-s))^{n-k}}{(n-k)!}$.

Theorem 2.2. *The counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ associated with a simple Poisson point process $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is strongly Markov.*

Proposition 2.3. *A simple counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ is associated with a* ***homogeneous Poisson process*** *with a constant intensity density λ, iff the inter-arrival time sequence $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ are* i.i.d. *random variables with an exponential distribution of rate λ.*

Proof. Let N_t be a counting process associated with a homogeneous Poisson point process on half-line with constant intensity density λ. From equivalence *iii_* in Theorem **??**, we obtain for any positive integer t,

$$P\{X_1 > t\} = P\{N_t = 0\} = e^{-\lambda t}.$$

It suffices to show that inter-arrivals time sequence $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is *i.i.d.* . We can show that N is Markov process with strong Markov property. Since the sequence of ordered points $\tilde{S} : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is a sequence of stopping times for the counting process, it follows from the strong Markov property of this process that $(N_{\tilde{S}_n+t} - N_{\tilde{S}_n} : t \geqslant 0)$ is independent of $\sigma(N_s : s \leqslant \tilde{S}_n)$ and hence of $\tilde{S}_n$ and $N_{\tilde{S}_n}$. Further, we see that

$$X_{n+1} = \inf\left\{t > 0 : N_{\tilde{S}_n+t} - N_{\tilde{S}_n} = 1\right\}.$$

It follows that $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is an independent sequence. For homogeneous Poisson point process, we have $N_{\tilde{S}_n+t} - N_{\tilde{S}_n} = N_t$ in distribution, and hence X_{n+1} has same distribution as X_1 for each $n \in \mathbb{N}$.

For the given *i.i.d.* inter-arrival time sequence $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ distributed exponentially with rate λ, we define the nth arrival instant $\tilde{S}_n \triangleq \sum_{i=1}^{n} X_i$ for each $n \in \mathbb{N}$, and the number of arrivals in time duration $(0,t]$

as $N_t \triangleq \sum_{n\in\mathbb{N}} \mathbb{1}_{\{\tilde{S}_n \leqslant t\}}$ for all $t \in \mathbb{R}_+$. It follows that N_t is path wise non-decreasing, integer-valued, right continuous, and simple since $P\{X_1 \leqslant 0\} = 0$. Therefore, N is a simple counting process such that

$$P\{N_t = 0\} = P\{X_1 > t\} = e^{-\lambda t}.$$

It follows that the void probabilities are exponential and hence the random variable N_t is Poisson with parameter λt for all $t \in \mathbb{R}_+$. Hence, N is a counting process associated with a homogeneous Poisson process with the constant intensity density λ from the equivalence ii_- in Theorem **??**. □

For many proofs regarding Poisson processes, we partition the sample space with the disjoint events $\{N_t = n\}$ for $n \in \mathbb{Z}_+$. We need the following lemma that enables us to do that.

Lemma 2.4. *For any finite time $t > 0$, the number of points on the interval $(0, t]$ from a Poisson process is finite almost surely.*

Proof. By strong law of large numbers, we have $\lim_{n\to\infty} \frac{S_n}{n} = \mathbb{E}[X_1] = \frac{1}{\lambda}$ almost surely. Fix $t > 0$ and we define a sample space subset $M = \{\omega \in \Omega : N(\omega, t) = \infty\}$. For any $\omega \in M$, we have $S_n(\omega) \leqslant t$ for all $n \in \mathbb{N}$. This implies $\limsup_n \frac{S_n}{n} = 0$ and $\omega \notin \left\{\lim_n \frac{S_n}{n} = \frac{1}{\lambda}\right\}$. Hence, the probability measure for set M is zero. □

2.1 Distribution functions

Lemma 2.5. *The following are true for the nth arrival instant $\tilde{S}_n$ of the Poisson arrival process $\tilde{S} : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ with constant intensity density λ.*

(a) The moment generating function is $M_{\tilde{S}_n}(\theta) = \mathbb{E}[e^{\theta \tilde{S}_n}] = \frac{\lambda^n}{(\lambda-\theta)^n} \mathbb{1}_{\{\theta<\lambda\}} + \infty \mathbb{1}_{\{\theta\geqslant\lambda\}}$.

(b) The distribution function is $F_n(t) \triangleq P\{\tilde{S}_n \leqslant t\} = 1 - e^{-\lambda t} \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!}$.

(c) The density function is Gamma distributed with parameters n and λ. That is, $f_n(s) = \frac{\lambda(\lambda s)^{n-1}}{(n-1)!} e^{-\lambda s}$.

Corollary 2.6. *Consider the counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ associated with the Poisson arrival process $\tilde{S} : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ having constant intensity density λ. The following are true.*

(a) The relation between distribution of nth arrival instant and probability mass function for the counting process is given by $F_n(t) = \sum_{j\geqslant n} P_j(t)$.

(b) For each $t \in \mathbb{R}_+$, the probability mass function $P_{N_t} \in \mathcal{M}(\mathbb{Z}_+)$ for discrete random variable $N_t : \Omega \to \mathbb{Z}_+$ is given by $P_n(t) \triangleq P_{N_t}(n) = P\{N_t = n)\} = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$.

(c) The relation between distribution of nth arrival instant and the mean of the counting process is given by $\sum_{n\in\mathbb{N}} F_n(t) = \mathbb{E}N_t$.

(d) For each $t \in \mathbb{R}_+$, the mean $\mathbb{E}[N_t] = \lambda t$, explaining the rate parameter λ for the Poisson process.

Proof. We observe the inverse relationship $\{\tilde{S}_n \leqslant t\} = \{N_t \geqslant n\}$ for all $n \in \mathbb{Z}_+$ and $t \in \mathbb{R}_+$.

(a) The result follows by taking the probability on both sides of the inverse relationship, to get $F_n(t) = P\{\tilde{S}_n \leqslant t\} = P\{N_t \geqslant n\} = \sum_{j\geqslant n} P\{N_t = j\} = \sum_{j\geqslant n} P_j(t)$.

(b) The result follows from the explicit from for the distribution of $\tilde{S}_n$ and recognizing that $P_n(t) = F_n(t) - F_{n+1}(t)$.

(c) The result from the following observation $\sum_{n\in\mathbb{N}} F_n(t) = \mathbb{E}\sum_{n\in\mathbb{N}} \mathbb{1}_{\{N_t\geqslant n\}} = \sum_{n\in\mathbb{N}} P\{N_t \geqslant n\} = \mathbb{E}N_t$.

(d) The result follows by summing the distribution function of the nth arrivals, to get $\mathbb{E}N_t = \sum_{n\in\mathbb{N}} F_n(t) = e^{-\lambda t} \sum_{n\in\mathbb{N}} \sum_{k\geqslant n} \frac{(\lambda t)^k}{k!} = \lambda t e^{-\lambda t} \sum_{k\in\mathbb{N}} \frac{(\lambda t)^{k-1}}{(k-1)!} = \lambda t$.

□

Remark 3. A Poisson process is not a stationary process. That is, the finite dimensional distributions are not shift invariant. This is clear from looking at the first moment $\mathbb{E}N_t = \lambda t$, which is linearly increasing in time.