Tutorial 1

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- Why do we study measure theoretic probability?

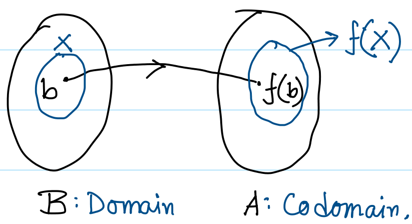
→ Because it is not easy to assign probabilities on the events from 'large' sample spaces.

→ One may find non-unique probability assignment to a single problem in non-mathematical terms.
ex- Bertrand Paradox.

- Functions.

Def^n (Function). Let A, B be sets. A function $f: B \to A$ is a rule that associates each element of B to a unique element of A.

- $f(b)$ is image of b, b is preimage of $f(b)$
- $f(X) = \{ f(b) : b \in X \}$ is image of X.



Range: $f(B) = \{ f(b) : b \in B \}$

- Not necessarily $f(B) = A$. If $f(B) = A$, then f is surjective.
- If each $a \in f(B)$ has unique preimage in B, then f is injective.
- If f is both injective and surjective, f is bijective.

- For every $a \in f(B)$ we can denote its preimage as $f^{-1}(a)$ such that $f(f^{-1}(a)) = a$
- Mind that $f^{-1}(a)$ is a set. If f is injective, $f^{-1}(a)$ is singleton set $\forall\, a \in f(B)$ and then only we can present $f^{-1}: f(B) \to B$ as a function. Still it is not exact inverse of $f: B \to A$. Surjection of f completes that.

- Cardinality:

· Two set are said to be equicardinal if $\exists$ a bijection between them. Let B is a set.

- B is countably infinite if $|B| = |\mathbb{N}|$
- B is finite if $|B| < |\mathbb{N}|$
- B is countable if $|B| \leqslant |\mathbb{N}|$

Ex 1. Let $(E_i : i \in \mathbb{N})$ be family of countably infinite sets. Then $\bigcup_{i \in \mathbb{N}} E_i$ is countable.

<u>Proof</u>: Let

$$E_1 = \{e_{11}, e_{12}, e_{13} \dots\}$$
$$E_2 = \{e_{21}, e_{22}, e_{23}, \dots\}$$
$$E_3 = \{e_{31}, e_{32}, e_{33} \dots\}$$
$$\vdots$$

Let h :

$$\begin{array}{cccc} e_{11} & e_{12} & e_{13} & \dots \\ e_{21} & e_{22} & e_{23} & \dots \\ e_{31} & e_{32} & e_{33} & \dots \\ \vdots & & & \end{array}$$

$$e_{ij} \to \frac{(i+j)(i+j-1)}{2} + i$$

Ex.2. Show that $[0,1]$ is uncountable.

<u>Proof</u>: Let us consider $[0,1]$ is countable. So we can so we should be able to enumerate $[0,1]$

Any element of $[0,1]$ can be written as infinite string of decimal digits i.e.

If $a \in [0,1]$ then, $a = 0.d_1 d_2 d_3 d_4 \dots$ where $d_i \in \{0,1,\dots 9\}$

$$D = \{(d_1 d_2 d_3 d_4 \dots) : d_i \in \{0,1,\dots,9\}\}$$

Let us assume the D is countable. Then we can enumerate D.

Let us write elements of D in an arbitrary enumeration rule.

$$D_1 = d_{11}\ d_{12}\ d_{13}\ d_{14} \dots$$
$$D_2 = d_{21}\ d_{22}\ d_{23}\ d_{24} \dots$$
$$D_3 = d_{31}\ d_{32}\ d_{33}\ d_{44} \dots$$

Let $f : \{0,\dots,9\} \to \{0,\dots 9\}$ s.t.

$$f(n) = \begin{cases} 3 & \text{if } n \neq 3 \\ 4 & \text{if } n = 4. \end{cases}$$

$D' = f(d_1)\, f(d_{22})\, f(d_{33})\, f(d_{44}) \cdots$ is not any where in the list.

So, $\mathcal{D}$ is uncountable.

· Each dyadic rational is mapped to 2 elements in $\mathcal{D}$

eg $\frac{1}{2} \to 0{\cdot}4999\ldots$, $\frac{1}{2} \to 0{\cdot}5000\ldots$ We need to show that the set of dyadic rational is countable. Let that is $\mathcal{D}_r$

$$\mathcal{D}_r \cup (\mathcal{D} \setminus \mathcal{D}_r) = \mathcal{D} \text{ is uncountable.}$$

$\mathcal{D}_r$ is countable then $\mathcal{D} \setminus \mathcal{D}_r$ must be uncountable. which has a bijection with $[0,1]$. $\boxtimes$

· Measure Theoretic Probability - Construction of sets on which we can assign probability.

· Sample Space: Set of all outcomes of a random experiment.

Denote by Ω (Universal set in set theory terminology)

· Subsets of Ω are called events (not all)

· Def^n (Algebra). A collection $\mathcal{F}_0$ (set of sets) of subsets of Ω is called an algebra if.

1. $\Omega \in \mathcal{F}_0$
2. If $A \in \mathcal{F}_0$, then $A^c \in \mathcal{F}_0$
3. If $A \in \mathcal{F}_0$, $B \in \mathcal{F}_0$ then $A \cup B \in \mathcal{F}_0$.

· We can show $\bigcup_{i=1}^{n} A_i \in \mathcal{F}_0$, $\bigcap_{i=1}^{n} A_i \in \mathcal{F}_0$ if $\{A_i\}_{i=1}^{n} \subseteq \mathcal{F}_0$

This is less than what we need. (will come back to it).

Def^n (σ-Algebra). A collection $\mathcal{F}$ of subsets of Ω is called an algebra if.

1. $\Omega \in \mathcal{F}$
2. If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$
3. If $A_i \in \mathcal{F}$, $i \in \mathbb{N}$, then $\bigcup_{i \in \mathbb{N}} A_i \in \mathcal{F}$

* For finite sample spaces all algebras are σ-algebra.

Ex.3. If $(A_n : n \in \mathbb{N})$ are subsets of $\mathcal{F}$ then

1. $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}_i$, 2. $\bigcup_{i=1}^{n} A_i \in \mathcal{F}$ 3. $\bigcap_{i=1}^{n} A_i \in \mathcal{F}$

Proof 1. $A_i \in \mathcal{F} \Rightarrow A_i^c \in \mathcal{F}$

$\therefore \bigcup_{i\in \mathbb{N}} A_i^c \in \mathcal{F} \Rightarrow \left(\bigcap_{i\in\mathbb{N}} A_i\right)^c \in \mathcal{F}$

$\Rightarrow \bigcup_{i\in\mathbb{N}} A_i \in \mathcal{F}$

2. We know that if $E_i \in \mathcal{F}$ $\forall i \in \mathbb{N}$ then $\bigcup_{i\in\mathbb{N}} E_i \in \mathcal{F}$.

We also know, $\phi \in \mathcal{F}$. Let $E_i = A_i$ for $i \in [n]$ and $E_i = \phi$ $\forall i > n$

$\therefore \bigcup_{i\in\mathbb{N}} E_i = \left(\bigcup_{i=1}^{n} A_i\right) \cup \left(\bigcup_{i>n} \phi\right) = \bigcup_{i=1}^{n} A_i$.

3. You can do using similar technique.

* Any σ-algebra is algebra.

· Measure

· The concept of σ-algebra is used in measure theory, to set are to be assigned a measure. So we call $(\Omega, \mathcal{F})$ a measurable space and $A \in \mathcal{F}$ as $\mathcal{F}$-measurable set.

Def^n: A measure is a function $\mu : \mathcal{F} \to [0,\infty]$ s.t.

1. $\mu(\phi) = 0$
2. If $(A_n : n \in \mathbb{N})$ is a collection of disjoint $\mathcal{F}$-measurable sets then

$$\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} \mu(A_i)$$

· $(\Omega, \mathcal{F}, \mu)$ is called a measure space.

Def^n A probability measure P on $(\Omega, \mathcal{F})$ is a function $P : \mathcal{F} \to [0,1]$ s.t.

1. $P(\phi) = 0$
2. If $(A_n : n \in \mathbb{N})$ is a collection of disjoint $\mathcal{F}$-measurable sets then

$$P\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} P(A_i)$$

3. $P(\Omega) = 1$.

· We call $(\Omega, \mathcal{F}, P)$ a probability space.

Ex 4. If $A_i \in \mathcal{F}$ $\forall i \in [n]$ for some $n \in \mathbb{N}$. then

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k) + \cdots + (-1)^{n-1} \mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right).$$

Proof: We will prove using mathematic induction.

Base case: $\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2)$

Induction hypothesis: $\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j)$

$$+ \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k) + \cdots + (-1)^{n-1} \mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right)$$

Induction step:

$$\mathbb{P}\left(\bigcup_{i=1}^{n+1} A_i\right) = \mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) + \mathbb{P}(A_{n+1}) - \mathbb{P}\left(\bigcup_{i=1}^{n} (A_i \cap A_{n+1})\right)$$

$$= \sum_{i=1}^{n+1} \mathbb{P}(A_i) - \sum_{\substack{i,j=1 \\ i<j}}^{n} \mathbb{P}(A_i \cap A_j) + \sum_{\substack{i,j,k=1 \\ i<j<k}}^{n} \mathbb{P}(A_i \cap A_j \cap A_k) + \cdots +$$

$$(-1)^{n-1} \mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) - \sum_{i=1}^{n} \mathbb{P}(A_i \cap A_{n+1}) + \sum_{\substack{i=1 \\ i<j}}^{n} \mathbb{P}(A_i \cap A_j \cap A_{n+1})$$

$$+ (-1)^{n-1} \sum_{j=1}^{n} \mathbb{P}\left(\bigcap_{\substack{i=1 \\ i \neq j}}^{n} (A_i \cap A_{n+1})\right) \qquad + (-1)^{n} \mathbb{P}\left(\bigcap_{i=1}^{n+1} A_i\right)$$

Assigning measures to Uncountable sample spaces:

- Let us assign a uniform measure on the sample space $\Omega = [0, 1]$

Can we assign probabilities to every set in 2^{Ω}?

Ans No.

1. If we assign non-zero probability to each singleton, Probability of infinite sets will be unbounded.
2. If we assign zero probability to each singleton, probability of any interval is undefined.

Generated σ-algebra: σ-algebra generated by a collection of subsets $\mathcal{C}$ of Ω is the smallest σ-algebra that contains $\mathcal{C}$.

Ex 5. If $\mathcal{C}$ is an arbitrary collection of subsets of Ω and $\mathcal{H}$ is any σ-algebra such that $\mathcal{C} \subseteq \mathcal{H}$ then there exists σ-algebra $\sigma(\mathcal{C}) \subseteq \mathcal{H}$

$\sigma(\mathcal{C}) \subseteq \mathcal{H}$.

Proof: Let $\{\mathcal{F}_\alpha : \alpha \in A\}$ denote the collection of all σ-algebras that contain $\mathcal{C}$.

$\{\mathcal{F}_\alpha : \alpha \in A\}$ is not empty as $2^\Omega \in \{\mathcal{F}_\alpha : \alpha \in A\}$

Then $\mathcal{C} \subseteq \bigcap_{\alpha \in A} \mathcal{F}_\alpha$ is a σ-algebra. (Need to show)

Also if $\mathcal{C} \subseteq \mathcal{H}$ which is a σ-algebra then $\mathcal{H} \in \{\mathcal{F}_\alpha : \alpha \in A\}$ and $\bigcap_{\alpha \in A} \mathcal{F}_\alpha \subseteq \mathcal{H}$. This is true for any $\mathcal{H}$.

Ex.6. $(\mathcal{F}_i,\ i \in I)$ is a collection of σ-algebras. Prove that $\bigcap_{i \in I} \mathcal{F}_i$ is a σ-algebra.

Proof: $\phi \in \bigcap_{i \in I} \mathcal{F}_i$ as $\phi \in \mathcal{F}_i \ \forall i \in I$.

Let $A \in \bigcap_{i \in I} \mathcal{F}_i \Rightarrow A \in \mathcal{F}_i \ \forall i \in I$

$\Rightarrow A^c \in \mathcal{F}_i \ \forall i \in I$

$\Rightarrow A^c \in \bigcap_{i \in I} \mathcal{F}_i$

Let $A_n \in \bigcap_{i \in I} \mathcal{F}_i \quad \forall n \in \mathbb{N}. \Rightarrow A_n \in \mathcal{F}_i \ \forall i \in I, \forall n \in \mathbb{N}$

$\Rightarrow \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{F}_i \ \forall i \in I$

$\Rightarrow \bigcup_{n \in \mathbb{N}} A_n \in \bigcap_{i \in I} \mathcal{F}_i$

So we showed all three condition of σ-algebra for $\bigcap_{i \in I} \mathcal{F}_i$. $\square$

Borel σ-algebra:

$\mathcal{B}(\mathbb{R})$ is the σ-algebra generated by collection of half closed intervals. Let $\mathcal{C} := \{(-\infty, x] : x \in \mathbb{R}\}$, then

$\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C})$.

Ex.6. Singleton sets are Borel sets.

Proof: For arbitrary b, $(-\infty, b] \in \mathcal{B}(\mathbb{R})$

Then $(-\infty, b - \frac{1}{n}] \in \mathcal{B}(\mathbb{R})$

$\Rightarrow (b - \frac{1}{n}, \infty) \in \mathcal{B}(\mathbb{R})$

$\Rightarrow \bigcap_{n \in \mathbb{N}} (b - \frac{1}{n}, \infty) \in \mathcal{B}(\mathbb{R}) \Rightarrow [b, \infty) \in \mathcal{B}(\mathbb{R})$

$\therefore (-\infty, b] \cap [b, \infty) = \{b\} \in \mathcal{B}(\mathbb{R})$

· Limit of Sets

Definition: Let $(A_n : n \in \mathbb{N})$ be a sequence of non-decreasing sets.

$$\lim_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} A_n$$

If $(A_n : n \in \mathbb{N})$ is a sequence of non-increasing sets

$$\lim_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} A_n$$

Definition: Let $(A_n : n \in \mathbb{N})$ be a sequence of sets

$$\limsup_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} A_m$$

$$\liminf_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} A_m$$

Lemma: For a sequence of sets $(A_n : n \in \mathbb{N})$, we have

$$\liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n$$

Proof: Let $E_n = \bigcup_{m\geq n} A_m$ $\therefore$ $(E_n : n \in \mathbb{N})$ is sequence of decreasing sets.

Let $F_n = \bigcap_{m\geq n} A_m$ $\therefore$ $(F_n : n \in \mathbb{N})$ is sequence of increasing sets.

Also, $F_1, F_2, \ldots F_n \subseteq A_n$ and $F_m \subseteq A_m$ $\forall\ m \geq n$.

$$\therefore \bigcup_{i\in\mathbb{N}} F_i \subseteq \bigcup_{m\geq n} A_m \quad \forall\ n \in \mathbb{N}$$

$$\Rightarrow \bigcup_{i\in\mathbb{N}} F_i \subseteq \bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} A_m$$

Continuity of Probability: Let $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of events in $\mathcal{F}$. such that $\lim_{n\to\infty} A_n$ exists. We have

$$\mathbb{P}\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} \mathbb{P}(A_n)$$

Proof:

$E_n = \bigcup_{m\geq n} A_m \in \mathcal{F} : n \in \mathbb{N}$ $\quad$ $F_n = \bigcap_{m\geq n} A_m \in \mathcal{F} : n \in \mathbb{N}$

↳ non-increasing set $\quad$ ↳ non decresing set.

$A_m \subseteq E_n \quad \forall\, m \geq n$

$F_n \subseteq A_m \quad \forall\, m \geq n$

$P(E_n) \geq \sup_{m \geq n} P(A_m)$

$P(F_n) \leq \inf_{m \geq n} P(A_m)$

$\lim_{n\to\infty} P(E_n) = P(\lim_{n\to\infty} E_n) = P\left(\bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} A_m\right) \geq \limsup_n \mathbb{P}(A_n)$

$\lim_{n\to\infty} P(F_n) = P(\lim_{n\to\infty} F_n) = P\left(\bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} A_m\right) \leq \liminf_n \mathbb{P}(A_n)$

$\limsup P(A_m) \leq P(\lim_{n\to\infty} A_n) \leq \liminf P(A_m)$

$$\boxed{\begin{aligned} &\lim_{n\to\infty}\sup_{m\geq n} P(A_m) = \limsup_{n\to\infty} \mathbb{P}(A_n) \\ \text{and } &\lim_{n\to\infty}\inf_{m\geq n} \mathbb{P}(A_m) = \liminf_{n\to\infty} \mathbb{P}(A_n). \end{aligned}}$$

Also, $\liminf \mathbb{P}(A_n) \leq \limsup (A_n)$

$\Rightarrow \limsup_n \mathbb{P}(A_n) = \liminf_n \mathbb{P}(A_n)$

$= \lim_n \mathbb{P}(A_n)$

$\lim_n \mathbb{P}(A_n) = \mathbb{P}(\lim_n A_n) \qquad \square$