Tutorial 2

Monday, 8 August 2022 10:18 PM

- $(\Omega, \mathcal{F}, \mathbb{P}) \rightarrow$ probability space

Sample space Event space probability measure.

<u>Independence of Events</u> For probability space $(\Omega, \mathcal{F}, P)$, a family of events $(A_i : i \in I)$ is said to be independent if for any finite set $F \subseteq I$, we have

$$P\left(\bigcap_{i\in F} A_i\right) = \prod_{i \in F} P(A_i)$$

* Let A, B, C are 3 events. Their Independence is assured by 4 equalities.

$$P(A \cap B) = P(A)P(B), \quad P(B \cap C) = P(B) \cdot P(C)$$
$$P(C \cap A) = P(C) \cdot P(A), \quad P(A \cap B \cap C) = P(A)P(B) \cdot P(C).$$

<u>Examples:</u> 1. Consider two independent tosses of a fair coin and the following events.

$H_1 = \{$ 1st toss is head $\} = \{HH, HT\}$

$H_2 = \{$ 2nd toss is head $\} = \{HH, TH\}$

$D = \{$ 1st and 2nd toss have diff. outcome $\}$

$= \{HT, TH\}$.

$P(H_1 \cap H_2) = P(\{HH\}) = \frac{1}{4}$ $\quad P(H_1) \cdot P(H_2) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

$P(H_1 \cap D) = P(\{HT\}) = \frac{1}{4}$ $\quad P(H_1) \cdot P(D) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$P(H_2 \cap D) = P(\{TH\}) = \frac{1}{4}$ $\quad P(H_2) \cdot P(D) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$P(H_1 \cap H_2 \cap D) = P(\phi) = 0$ $\quad P(H_1) \cdot P(H_2) \cdot P(D) = \frac{1}{8}$

2. Consider two independent rolls of a fair 6-sided die, and the following events:

$A = \{$ 1st roll is $1, 2, 3 \} = \{(x, y) : x \in [3], y \in [6]\}$

$B = \{$ 2nd roll is $4, 5, 6 \} = \{(x, y) : x \in [6], y \in [3]\}$

$C = \{$ the sum of the two rolls is $9\}$

$= \{(x, y) : x + y = 9, x \in [6], y \in [6]\}$

$P(A) = \frac{1}{2}$, $P(B) = \frac{1}{2}$, $P(C) = \frac{1}{9} \Rightarrow P(A)P(B) \cdot P(C) = \frac{1}{36}$.

$P(A \cap B \cap C) = P(\{(3,6)\}) = \frac{1}{36}$

$P(A \cap B) = P(A) \cdot P(B)$. $\quad P(A \cap C) = \frac{1}{36}$ $\quad P(A)P(C) = \frac{1}{2} \cdot \frac{1}{9} = \frac{1}{18}$

$P(B \cap C) =$ [illegible]

$P(B \cap C) = \frac{1}{12}$ $\qquad P(B)\cdot P(C) = \frac{1}{2}\cdot\frac{1}{9} = \frac{1}{18}$

Independence of σ-algebras Let $\mathcal{F}_i, i \in I$ are independent σ-algebras $(\mathcal{F}_i \subseteq \mathcal{F})$ if for any collection of events $(A_i \in \mathcal{F}_i : i \in F)$, with finite set $F \subseteq I$

$$P\left(\bigcap_{i\in F} A_i\right) = \prod_{i\in F} P(A_i)$$

Random Variable : Consider a probability space $(\Omega, \mathcal{F}, P)$. A random variable $X : \Omega \to \mathbb{R}$ is a $\mathbb{R}$-valued function from the sample space to real numbers, s.t for each $x \in \mathbb{R}$ the event

$$X^{-1}(-\infty, x] = A_X(x) \triangleq \{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{F}$$

That is $X^{-1}(B_x)$ are $\mathcal{F}$-measurable sets where $B_x = (-\infty, x]\ \forall x \in \mathbb{R}$.
– X is called $\mathcal{F}$-measurable random variable.

* $(B_x : x \in \mathbb{R})$ is the generating collection for Borel σ-algebra

Ex 1. $B \in \mathcal{B}(\mathbb{R})$. Show that $X^{-1}(B) \in \mathcal{F}$ if X is a $\mathcal{F}$-measurable r.v.

Sol^n. We have that the generating sets, $B_x = (-\infty, x] \in \mathcal{B}(\mathbb{R})$ have preimages in $\mathcal{F}$. We need to show that any set $B \in \mathcal{B}(\mathbb{R})$ has its preimage in $\mathcal{F}$.

It suffices to show that

① $X^{-1}(B^c) = X^{-1}(B)^c$ where $B \in \mathcal{B}(\mathbb{R})$ and $X^{-1}(B) \in \mathcal{F}$

② $X^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right) = \bigcup_{i=1}^{\infty} X^{-1}(B_i)$ where $B_i \in \mathcal{B}(\mathbb{R})$ and $X^{-1}(B_i) \in \mathcal{F}\ \forall i \in \mathbb{N}$

1. Let $\omega \in X^{-1}(B^c) \Rightarrow X(\omega) \in B^c$
$\Rightarrow X(\omega) \notin B$
$\Rightarrow \omega \notin X^{-1}(B)$
$\Rightarrow \omega \in X^{-1}(B)^c$

$\therefore X^{-1}(B^c) \subseteq X^{-1}(B)^c$

Let $\omega \in X^{-1}(B)^c \Rightarrow \omega \notin X^{-1}(B)$
$\Rightarrow X(\omega) \notin B$
$\Rightarrow X(\omega) \in B^c$
$\Rightarrow \omega \in X^{-1}(B^c)$

$\therefore X^{-1}(B)^c \subseteq X^{-1}(B^c)$ — B

From A and B, $X^{-1}(B)^c = X^{-1}(B^c)$

2. Let $\omega \in X^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right)$

$\Rightarrow X(\omega) \in \bigcup_{i=1}^{\infty} B_i$

$\Rightarrow X(\omega) \in B_i$ for some $i \in \mathbb{N}$

$\Rightarrow \omega \in X^{-1}(B_i)$ for some $i \in \mathbb{N}$

$\Rightarrow \omega \in \bigcup_{i=1}^{\infty} X^{-1}(B_i)$

$\therefore X^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right) \subseteq \bigcup_{i=1}^{\infty} X^{-1}(B_i)$ — C

Let $\omega \in \bigcup_{i=1}^{\infty} X^{-1}(B_i)$

$\Rightarrow \omega \in X^{-1}(B_i)$ for some $i \in \mathbb{N}$

$\Rightarrow X(\omega) \in B_i$ for some $i \in \mathbb{N}$

$\Rightarrow X(\omega) \in \bigcup_{i=1}^{\infty} B_i$

$\Rightarrow \omega \in X^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right)$

$\therefore \bigcup_{i=1}^{\infty} X^{-1}(B_i) \subseteq X^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right)$ — D

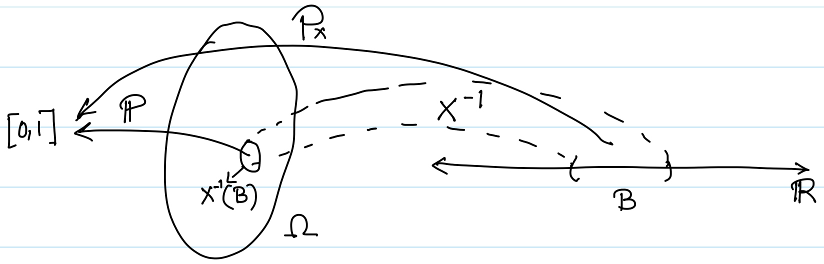
From C and D, $\bigcup_{i=1}^{\infty} X^{-1}(B_i) = X^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right)$ $\square$

- Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be two measurable spaces. Let $f: X \to Y$. f is called $\mathcal{X}$-measurable if for every $B \in \mathcal{Y}$, $f^{-1}(B) \in \mathcal{X}$.

<u>$\mathcal{F}$-measurable $\mathbb{R}$-valued function</u> : A function $f: \Omega \to \mathbb{R}$ is called $\mathcal{F}$-measurable if for any $B \in \mathcal{B}(\mathbb{R})$, $f^{-1}(B) \in \mathcal{F}$.

<u>Probability law induced by random variable X</u>: Probability law of X, $\mathbb{P}_X : \mathcal{B}(\mathbb{R}) \to [0,1]$ is defined by

$$\mathbb{P}_X(B) = \mathbb{P}(X^{-1}(B)) \quad \forall\ B \in \mathcal{B}(\mathbb{R})$$
$$= \mathbb{P}(\{\omega \in \Omega : X(\omega) \in B\})$$



Ex.2. $(\mathbb{R}, \mathcal{B}(\mathbb{R}), P_X)$ is a probability space for any r.v. X.

Sol^n Given $\mathbb{R}$ be the sample space $\mathcal{B}(\mathbb{R})$ is a σ-algebra on the Sample space. So $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is indeed a measurable space. It remains to check that P_X is a probability measure.

1. $P_X(\phi) = \mathbb{P}(X^{-1}(\phi)) = \mathbb{P}(\phi) = 0$
2. $P_X(\mathbb{R}) = \mathbb{P}(X^{-1}(\mathbb{R})) = \mathbb{P}(\Omega) = 1$.
3. Let $(B_i, i \in \mathbb{N})$ is a collection of disjoint sets in $\mathcal{B}(\mathbb{R})$

Claim: $(X^{-1}(B_i), i \in \mathbb{N})$ are also disjoint.

Let $\omega \in X^{-1}(B_i)$ and $\omega \in X^{-1}(B_j)$ for $i \neq j$

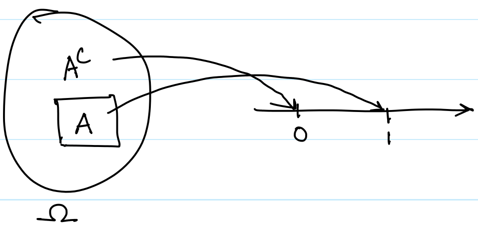
$\Rightarrow X(\omega) \in B_i$ and $X(\omega) \in B_j$ but $B_i \cap B_j = \phi$

hence $\nexists\ \omega \in X^{-1}(B_i)$ and $\omega \in X^{-1}(B_j)$ for $i \neq j$

So $(X^{-1}(B_i), i \in \mathbb{N})$ are disjoint sets.

$$P_X\Big(\bigcup_{i\in\mathbb{N}} B_i\Big) = \mathbb{P}\Big(X^{-1}\Big(\bigcup_{i\in\mathbb{N}} B_i\Big)\Big)$$
$$= \mathbb{P}\Big(\bigcup_{i\in\mathbb{N}} X^{-1}(B_i)\Big) \qquad [\text{Ex } 1]$$
$$= \sum_{i\in\mathbb{N}} \mathbb{P}(X^{-1}(B_i)) \qquad [\text{as } \mathbb{P} \text{ is a prob. meas.}]$$
$$= \sum_{i\in\mathbb{N}} P_X(B_i) \qquad \square$$

Indicator Random Variable: $\mathbb{1}_A : \Omega \to \{0,1\}$ is call an indicator r.v. if
$$\mathbb{1}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$



Take a Borel set $B \in \mathcal{B}(\mathbb{R})$
$$\mathbb{1}_A^{-1}(B) = \begin{cases} \phi \text{ if } 0 \notin B, 1 \notin B \\ A \text{ if } 0 \notin B, 1 \in B \\ A^c \text{ if } 0 \in B, 1 \notin B \\ \Omega \text{ if } 0 \in B, 1 \in B \end{cases}$$

or,
$$\mathbb{1}_A^{-1}(-\infty, x] = \begin{cases} \phi & \text{if } x < 0 \\ A^c & \text{if } x \in [0,1) \\ \Omega & \text{if } x \geq 1 \end{cases}$$

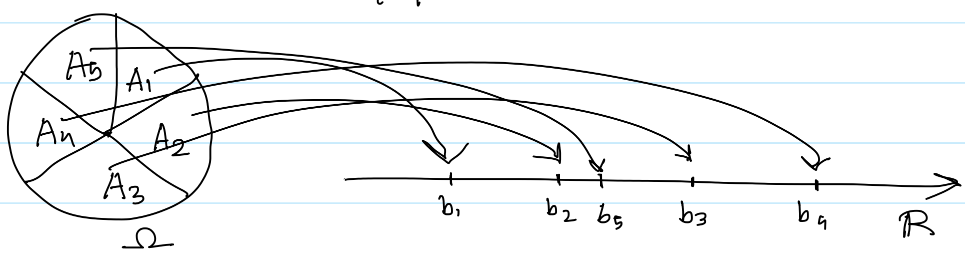
$$F_{\mathbb{1}_A}(x) = P_{\mathbb{1}_A}((-\infty, x]) = \mathbb{P} \circ \mathbb{1}_A^{-1}((\infty, x])$$
$$= \begin{cases} \mathbb{P}(\phi) & \text{if } x < 0 \\ \mathbb{P}(A^c) & \text{if } x \in [0,1) \\ \mathbb{P}(\Omega) & \text{if } x \geq 1 \end{cases} = \begin{cases} 0 & \text{if } x < 0 \\ 1 - \mathbb{P}(A) & \text{if } x \in [0,1) \\ 1 & \text{if } x \geq 1 \end{cases}$$

So, $\mathbb{1}_A$ is $\underbrace{\{\phi, A, A^c, \Omega\}}_{=:\mathcal{F}}$-measurable.

Let $\mathcal{F} \subseteq \mathcal{G}$. $\mathbb{1}_A$ is $\mathcal{G}$-measurable too.

Simple Random Variable : $X : \Omega \to \mathcal{X} \subseteq \mathbb{R}$ is called simple r.v. If $|\mathcal{X}| = n$ for some $n \in \mathbb{N}$. Let $\mathcal{X} = \{b_1, \dots, b_n\}$

$$X(\omega) = \sum_{i=1}^{n} b_i \cdot \mathbb{1}_{A_i}(\omega) \qquad \mathbb{R}, \mathcal{B}(\mathbb{R}), P_X.$$



σ-algebra generated by a random variable : Let $X : \Omega \to \mathbb{R}$ be an $\mathcal{F}$-measurable random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$. The smallest event space generated by $A_X(x) = X^{-1}(B_x) = X^{-1}(-\infty, x]$ for $x \in \mathbb{R}$ is called σ-algebra generated by X.

$$\sigma(X) = \sigma(\{A_X(x) : x \in \mathbb{R}\})$$

· Prove that $\sigma(X) := \{X^{-1}(B) : B \in \mathcal{B}(\mathbb{R})\}$

Ex 3. Show that if a simple random variable $X = \sum_{i=1}^{n} b_i \mathbb{1}_{A_i}$ then
$$\sigma(X) = \sigma(\{A_i : i \in [n]\}).$$

Solⁿ : Let $A \in \sigma(X) \Rightarrow X(A) = B \in \mathcal{B}(\mathbb{R})$.

If $A \neq \phi$, $B \in \{b_i : i \in [n]\}$

$$\Rightarrow A = \bigcup_{b \in B} X^{-1}(b) = \bigcup_{\substack{i \in [n] \\ \text{s.t. } b_i \in B}} A_i \in \sigma(\{A_i : i \in [n]\})$$

If $A = \phi$, $A \in \sigma(\{A_i : i \in [n]\}) \Rightarrow \sigma(X) \subseteq \sigma(\{A_i : i \in [n]\})$ — A

$\{b_i\}$'s are singleton Borel sets.

$\therefore X^{-1}\{b_i\} = A_i \in \sigma(X) \ \forall i \in [n]$

$\Rightarrow \{A_i : i \in [n]\} \subseteq \sigma(X)$

$\Rightarrow \sigma\{A_i : i \in [n]\} \subseteq \sigma(X)$ — B.

From A & B. result follows $\square$

* σ-algebra generated by random variables are independent

$\Leftrightarrow$ random variables are independent. $F_{XY}(x,y) = F_X(x) F_Y(y)$

<u>Ex 4</u> $P(X=x) = F(x) - \lim_{y \uparrow x} F(y)$.

<u>Sol^n</u>. $\{\omega : X(\omega) = x\} = \bigcap_{n\in\mathbb{N}} \{\omega : x - \frac{1}{n} < X(\omega) \leq x\}$

$= \lim_{n\to\infty} \{\omega : x - \frac{1}{n} < X(\omega) \leq x\}$

$$\begin{aligned}
\therefore P(X=x) &= P(\{\omega : X(\omega) = x\}) \\
&= P\left(\lim_{n\to\infty} \{\omega : x - \tfrac{1}{n} < X(\omega) \leq x\}\right) \\
&= \lim_{n\to\infty} P(\{\omega : x - \tfrac{1}{n} < X(\omega) \leq x\}) \\
&= \lim_{n\to\infty} P(\{\omega : X(\omega) \leq x\} \setminus \{\omega : X(\omega) \leq x - \tfrac{1}{n}\}) \\
&= F_X(x) - \lim_{n\to\infty} F_X(x - \tfrac{1}{n}) \\
&= F_X(x) - \lim_{y \uparrow x} F_X(y) \qquad \square
\end{aligned}$$

<u>Law of total probability</u>: If $(B_n : n \in \mathbb{N})$ is partition of Ω, i.e., $B_n \cap B_m = \phi \;\forall n \neq m,\; n, m \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} B_n = \Omega$, then for any $A \in \mathcal{F}$

$$P(A) = \sum_{n\in\mathbb{N}} P(A \cap B_n)$$

<u>Conditional Probability</u>: Conditional probability of an event A given another event B s.t $P(B) > 0$ is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

From the def^n of conditional probability -

- $P(A \cap B) = P(A|B) \cdot P(A)$

From the above equation and Law of total probability -

- $P(A) = \sum_{n\in I} P(A|B_n) \cdot P(B_n)$ if $P(B_n) > 0$. $\forall n$

From Conditional probability and the above equation -

- $P(B_i|A) = \dfrac{P(B_i \cap A)}{P(A)} = \dfrac{P(A|B_i) \cdot P(B_i)}{\sum_{n\in I} P(A|B_n) \cdot P(B_n)}$ if $P(A) > 0$.

- For events $A_1, A_2, \ldots, A_n$ satisfying $P(A_1 \cap A_2 \cap \cdots \cap A_{n-1}) > 0$, prove that

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)\cdot P(A_2 \mid A_1)\cdot \cdots P(A_n \mid A_1 \cap A_2 \cdots \cap A_{n-1})$$

(multiplication rule)

Ex-5. There are n urns of which the r^{th} urn contains $r-1$ red balls an $n-r$ magenta balls. You pick an urn at random and remove two balls at random w/o replacement. Find the probability that

a. The 2^{nd} ball is magenta.

b. The 2^{nd} ball is magenta given the first is magenta.

c. Given the first ball is magenta what is the probability that i^{th} urn was chosen?

Sol^n. Let the r^{th} urn is picked $\equiv$ event B_r

Chosen an urn, the first draw could be red or magenta

Let $D_r^1 \equiv$ The first draw is magenta.

$D_m^1 \equiv$ The first draw is red.

Let $D_m^2 \equiv$ The 2nd draw is magenta.

We need to find $P(D_m^2)$.

$\bigcup_{r=1}^{n} B_r = \Omega$. Using law of total probability

$$P(D_m^2) = \sum_{i=1}^{n} P(D_m^2 \cap B_i) = \sum_{=1}^{n} P(D_m^2 \mid B_i)\cdot P(B_i)$$

Also, $D_m^1 \cup D_r^1 = \Omega$, So, for fixed r, using law of total probability we get

$$\begin{aligned}
P(D_m^2 \mid B_i) &= P(D_m^1 \cap D_m^2 \mid B_i) + P(D_r^1 \cap D_m^2 \mid B_i) \\
&= P(D_m^1 \mid B_i)\, P(D_m^2 \mid B_i \cap D_m^1) \\
&\quad + P(D_r^1 \mid B_i)\cdot P(D_m^2 \mid B_i \cap D_r^1) \\
&= \frac{n-i}{n-1}\cdot\frac{n-i-1}{n-2} + \frac{i-1}{n-1}\cdot\frac{n-i}{n-2} \\
&= \frac{n-i}{n-1}
\end{aligned}$$

$$\begin{aligned}
P(D_m^2) &= \sum_{i=1}^{n} P(B_i)\, P(D_m^2 \mid B_i) \\
&= \sum_{i=1}^{n} \frac{1}{n}\cdot\frac{n-i}{n-1} = \frac{n}{n-1} - \frac{\cancel{n}\cdot(n+1)}{2\cdot\cancel{n}\,(n-1)} \\
&= \frac{2n-n-1}{2\cdot(n-1)} \\
&= \frac{n-1}{2(n-1)} = \frac{1}{2} \quad \text{Ans.}
\end{aligned}$$

$$P(D_m^2 \mid D_m^1) = \frac{\sum_{i=1}^{n} P(D_m^1 \cap D_m^2 \cap B_i)}{\sum_{l=1}^{n} P(D_m^1 \cap B_i)}$$

$$= \frac{\sum_{i=1}^{n-2} P(D_m^1 \cap D_m^2 \cap B_i)}{\sum_{l=1}^{n-1} P(D_m^1 \cap B_i)}$$

$$= \frac{\sum_{i=1}^{n-2} \cdot P(D_m^2 \mid D_m^1 \cap B_i) \cdot P(D_m^1 \mid B_i) \cdot P(B_i)}{\sum_{l=1}^{n-1} P(D_m^1 \mid B_i) \cdot P(B_i)}$$

$$= \frac{\sum_{i=1}^{n-2} \frac{n-i-1}{n-2} \cdot \frac{n-i}{n-1} \cdot \frac{1}{n}}{\sum_{l=1}^{n-1} \frac{n-l}{n-1} \cdot \frac{1}{n}}$$

$$= \frac{n(n-1)}{n(n-1)(n-2)} \frac{\sum_{i=1}^{n-2} n^2 - (2n-1)i + i^2 - n}{\sum_{l=1}^{n-1} n - l.}$$

$$= \frac{1}{n-2} \cdot \frac{n(n-1)(n-2) - \frac{(2n-1)(n-1)(n-2)}{2} + \frac{(n-2)(n-1)(2n-3)}{6}}{n^2 - \frac{(n-1)n}{2}}$$

$$= \frac{2}{n(n-1)(n-2)} \cdot \left(n(n-1)(n-2) - \frac{(2n-1)(n-1)(n-2)}{2} + \frac{(n-2)(n-1)(2n-3)}{6} \right)$$

$$= 2 - \frac{2n-1}{n} + \frac{2n-3}{3n} = 2 - 2 + \frac{1}{n} + \frac{2}{3} - \frac{1}{n}$$

$$= \frac{2}{3}$$

$$P(B_i \mid D_m^1) = \frac{P(D_m^1 \mid B_i) \cdot P(B_i)}{\sum_{l=1}^{n} P(D_m' \mid B_l) P(B_l)}$$

$$= \frac{\frac{n-i}{n-1} \cdot \frac{1}{n}}{\sum_{l=1}^{n} \cdot \frac{n-l}{n-1} \cdot \frac{1}{n}} = \frac{n-i}{\sum_{l=1}^{n} n-l} = \frac{n-i}{n^2 - \frac{n \cdot (n+1)}{2}}$$

$$= \frac{2(n-i)}{n(n-1)}$$