Tutorial 3

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* An n-dimensional random vector, $X : \Omega \to \mathbb{R}^n$ s.t.

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$$ where $X_i : \Omega \to \mathbb{R}$ $\forall i \in [n]$ are random variables.

<u>Independence of σ-algebras</u> Let $\mathcal{F}_i$, $i \in I$ are independent σ-algebras ($\mathcal{F}_i \subseteq \mathcal{F}$) if for any collection of events $(A_i \in \mathcal{F}_i : i \in F)$, with finite set $F \subseteq I$

$$P\left(\bigcap_{i\in F} A_i\right) = \prod_{i \in F} P(A_i)$$

<u>Independence of Random Variables</u> A collection of n random variables $X_i : \Omega \to \mathbb{R}$, $i \in [n]$ defined on $(\Omega, \mathcal{F}, P)$ are independent if

$$F_{X_1 \dots X_n}(x_1, \dots, x_n) = \prod_{i=1}^{n} F_{X_i}(x_i) \quad \forall \; x_i \in \mathbb{R},\ i \in [n].$$

$$\equiv$$

$$F_X(x) \qquad \text{where } x = (x_1, \dots, x_n).$$

<u>σ-algebra generated by a Random Variable</u> : Let Y be an RV on $(\Omega, \mathcal{F}, P)$. $A_Y(y) = \{ Y^{-1}(-\infty, y] : y \in \mathbb{R} \}$. The σ-algebra generated by Y,

$$\sigma(Y) = \sigma(\{ A_Y(y) : y \in \mathbb{R} \})$$

Let $(\Omega, \mathcal{F}, P)$ be the probability space.
Show that $\{X_i : i \in [n]\}$ is a collection of independent RVs if and only if $\{\sigma(X_i) : i \in [n]\}$ is a collection of independent σ-algebras.

Proof ($\Rightarrow$) Let $\{\sigma(X_i) : i \in [n]\}$ is a collection of independent σ-algebras. Let us choose arbitrary $x = (x_1, \cdots, x_n)$ and define

$$A_{X_1, \dots X_n}(x) = \bigcap_{i \in [n]} A_{X_i}(x_i) \quad \text{where} \quad A_{X_i}(x_i) \in \sigma(X_i) \ \forall \ i \in [n]$$

and $x_i \in \mathbb{R}$.

$$\therefore F_{X_1 \dots X_n}(x) = P(A_{X_1 \dots X_n}(x))$$

$$= P\left(\bigcap_{i \in [n]} A_{X_i}(x_i)\right)$$

$$= \prod_{i \in [n]} P(A_{X_i}(x_i))$$

$$= \prod_{i \in [n]} F_{X_i}(x_i)$$

($\Leftarrow$) Not in the scope of this course.

Joint CDF of two random variables: Let X and Y be two RVs defined on the probability space $(\Omega, \mathcal{F}, P)$.

$$F_{XY}(x, y) = P(\{\omega \mid X(\omega) \leq x, Y(\omega) \leq y\})$$

If X, Y are two independent RVs,

$$F_{XY}(x, y) = P(\{\omega \mid X(\omega) \leq x\}) \cdot P(\{\omega \mid Y(\omega) \leq y\})$$

$$= F_X(x) \cdot F_Y(y) \qquad \forall \ x, y \in \mathbb{R}$$

Ex. 1 Let F_{XY} is the joint distribution of RV X, Y on $(\Omega, \mathcal{F}, \mathbb{P})$. Show that

$$P(\{\omega : a < X(\omega) \leq b, c < Y(\omega) \leq d\})$$

$$= F_{XY}(b, d) - F_{XY}(a, d) - F_{XY}(b, c) + F_{XY}(a, c)$$

Sol^n $P(\{\omega : a < X(\omega) \leq b\} \cap \{\omega : c < Y(\omega) \leq d\}) =: P(C)$

Let $A = \{\omega : X(\omega) \leq b, c < Y(\omega) \leq d\}$

Let $A = \{\omega : X(\omega) \le b,\ c < Y(\omega) \le d\}$

$B = \{\omega : Y(\omega) \le d,\ a < X(\omega) \le b\}$

$$P(C) = P(A \cap B) = P(A) + P(B) - P(A \cap B)$$
$$= F_{XY}(b,d) - F_{XY}(b,c) + F_{XY}(b,d) - F_{XY}(a,d) - F_{XY}(b,d) + F_{XY}(a,c)$$
$$= F_{XY}(b,d) - F_{XY}(b,c) - F_{XY}(a,d) + F_{XY}(a,c) \qquad \square$$

Discret Random Variable: If a random variable takes value in some countable subset of $\mathbb{R}$.

Continuous Random Variable: X is called continuous RV if we can write

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\, du \qquad \text{for } f_X : \mathbb{R} \to [0,\infty)$$

<u>Ex 2</u> X and Y have joint distribution function

$$F_{XY}(x,y) = \begin{cases} 0 & \text{if } x < 0 \\ (1-e^{-x})\left(\frac{1}{2} + \frac{1}{\pi}\tan^{-1}(y)\right) & \text{if } x \ge 0 \end{cases}$$

<u>Sol^n</u>

$$\frac{\partial}{\partial x} F_{XY}(x,y) = \begin{cases} 0 & \text{if } x < 0 \\ e^{x}\left(\frac{1}{2} + \frac{1}{\pi}\tan^{-1}(y)\right) & \text{if } x \ge 0 \end{cases}$$

$$\frac{\partial^2}{\partial x \partial y} F_{XY}(x,y) = \begin{cases} 0 & \text{if } x < 0 \\ e^{x}/\pi(1+y^2) & \text{if } x \ge 0. \end{cases}$$

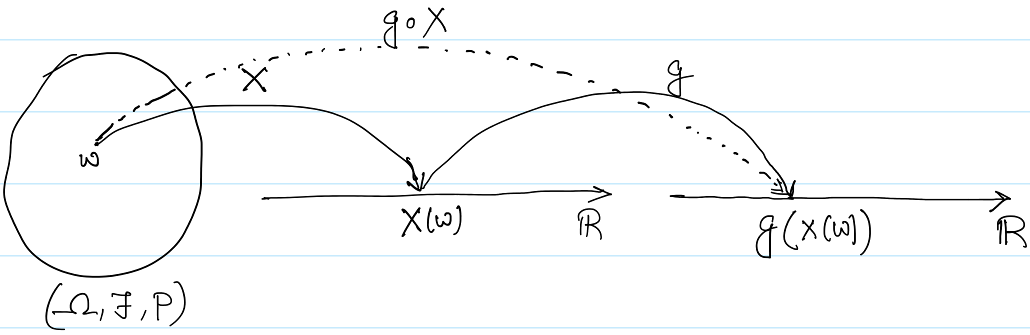
$$\left(= \frac{\partial^2}{\partial y \partial x} F_{XY}(x,y) \right)$$

$$\Rightarrow \quad F_{XY}(u,v) = \int_{-\infty}^{x} du \int_{-\infty}^{y} dv \left(\frac{\partial}{\partial u}\left(\frac{\partial}{\partial v} F_{XY}(u,v)\right)\right)$$

$$\therefore \quad f_{XY}(x,y) = \begin{cases} 0 & \text{if } x < 0 \\ e^{x}/\pi(1+y^2) & \text{if } x \ge 0 \end{cases}$$

$$\therefore \; f_{XY}(x,y) = \begin{cases} 0 & \text{if } x < 0 \\ \dfrac{e^{x}}{\pi(1+y^2)} & \text{if } x \geq 0 \end{cases}$$

Transformation of Random Variables



Q1. Is $g \circ X$ a RV? If yes, when?

Q2. How to find the probability law of $g \circ X$?

1. One sufficient condition is if g is Borel measurable function.
2. If g is monotonically increasing.

$$\begin{aligned} F_{g\circ X}(y) &= \mathbb{P}\left(\{\omega : g(X(\omega)) \leq y\}\right) \\ &= \mathbb{P}\left(\{\omega : X(\omega) \leq g^{-1}(y)\}\right) \\ &= F_X\left(g^{-1}(y)\right) \\ &= \int_{-\infty}^{g^{-1}(y)} f_X(x)\, dx \end{aligned}$$

As g is monotonically increasing function, $g(x) \leq y \Rightarrow x \leq g^{-1}(y)$

Let $x = g^{-1}(t)$ so $g'(x)\, dx = dt$

$$\therefore \; F_{g\circ X}(y) = \int_{-\infty}^{y} f_X\left(g^{-1}(t)\right) \cdot \frac{dt}{g'\left(g^{-1}(t)\right)}$$

$$\Rightarrow \; f_{g\circ X}(y) = f_X\left(g^{-1}(y)\right) \cdot \frac{1}{g'\left(g^{-1}(y)\right)}$$

If g is monotonically decreasing, $f_{g\circ X}(y) = f_X\left(g^{-1}(y)\right) \cdot \dfrac{-1}{g'\left(g^{-1}(y)\right)}$

For any monotonic function, g

$$f_{g\circ X}(y) = \frac{f_X(g^{-1}(y))}{|g'(g^{-1}(y))|}$$

Ex 3. Let X be a Gaussian RV of mean 0 and variance 1. i.e, $X \sim \mathcal{N}(0,1)$. Find the distribution of $Y = e^X$.

<u>Sol^n</u>. $g(x) = e^x$, $g'(x) = e^x$, $g^{-1}(y) = \ln(y)$, $g'(g^{-1}(y)) = y$

$$f_Y(y) = \frac{f_X(\ln(y))}{y} = \frac{1}{y\sqrt{2\pi}} e^{-\frac{(\ln y)^2}{2}} \quad \text{for } y > 0$$

Ex.4. Let $X_1, \dots, X_n$ be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with joint CDF $F_{X_1 \dots X_n}$. Define

$$Y = \min(X_1, \dots, X_n) \quad \text{and}$$
$$Z = \max(X_1, \dots, X_n)$$

a. Show that Y and Z are Random variables

b. Find distributions of Y & Z.

Sol^n. a. Consider the event

$$\{\omega \in \Omega : Z(\omega) \le z\} = \{\omega \in \Omega : X_1(\omega) \le z, \dots, X_n(\omega) \le z\}$$
$$= \bigcap_{i=1}^{n} \{\omega \in \Omega : X_i(\omega) \le z\} \in \mathcal{F}$$

$\Rightarrow$ Z is an RV.

$$\{\omega \in \Omega : Y(\omega) \le y\} = \{\omega \in \Omega : Y(\omega) > y\}^c$$
$$= \{\omega \in \Omega : X_1(\omega) > y, \dots, X_n(\omega) > y\}^c$$
$$= \left(\bigcap_{i=1}^{n} \{\omega \in \Omega : X_i(\omega) > y\}\right)^c$$
$$= \bigcup_{i=1}^{n} \{\omega \in \Omega : X_i(\omega) > y\}^c \in \mathcal{F}.$$

$\Rightarrow$ Y is an RV

b. $$F_Z(z) = P(\{\omega\in\Omega : Z(\omega)\le z\})$$
$$= P\left(\bigcap_{i=1}^{n}\{\omega\in\Omega : X_i(\omega)\le z\}\right)$$
$$= F_{X_1\cdots X_n}(z)$$

$$\bar{F}_Y(y) = P(\{\omega\in\Omega : Y(\omega) > y\})$$
$$= P\left(\bigcap_{i=1}^{n}\{\omega\in\Omega : X_i(\omega) > y\}\right)$$
$$= \bar{F}_{X_1\cdots X_n}(z).$$

· What if $X_1, \ldots, X_n$ iid random variables?

1. $$F_Z(z) = F_{X_1\cdots X_n}(z) = F_{X_1}(z)\cdots F_{X_n}(z)$$
$$= F_{X_1}(z)^n$$

2. $$\bar{F}_Y(y) = \bar{F}_{X_1\cdots X_n}(y) = \bar{F}_{X_1}(y)\cdots\bar{F}_{X_n}(y)$$
$$= \left(\bar{F}_{X_1}(y)\right)^n$$

$$F_Y(y) = P(\{\omega\in\Omega : Y(\omega)\le y\}) = P(\{\omega\in\Omega : Y(\omega) > y\}^c\})$$
$$= 1 - P(\{\omega\in\Omega : Y(\omega) > y\})$$
$$= 1 - \bar{F}_Y(y)$$

$$\therefore\ 1 - F_Y(y) = \left(1 - F_{X_1}(y)\right)^n$$
$$\Rightarrow\ F_Y(y) = 1 - \left(1 - F_{X_1}(y)\right)^n$$

Ex 5. Let $X_1, \ldots, X_n$ be indipendent exponentially distributed RVs with parameters $\lambda_1, \ldots, \lambda_n$. $F_{X_i}(x) = 1 - e^{-\lambda_i x}$ for $x > 0$. Let $Y = \min(X_1, \ldots, X_n)$. Show that Y is exponetially distributed with parameter $\lambda_1 + \cdots + \lambda_n$.

Sol^n $$F_Y(y) = 1 - \prod_{i=1}^{n}\left(1 - F_{X_i}(x)\right)$$

Sol^n

$$F_Y(y) = 1 - \prod_{i=1}^{n} \left(1 - F_{X_i}(x)\right)$$

$$= 1 - \prod_{i=1}^{n} e^{-\lambda_i x}$$

$$= 1 - e^{-\left(\sum_{i=1}^{n} \lambda_i\right) x}$$

$$\Rightarrow \quad Y \sim \mathrm{Exp}\left(\sum_{i=1}^{n} \lambda_i\right). \qquad \square$$