Tutorial 4

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<u>Exercise 1</u> Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We define a sequence of independent RVs $(X_i : \Omega \to \{0,1\}, i \in \mathbb{N})$ such that $\mathbb{P}$ induces probability law on X_i,

$$P_{X_i}(X_i = 1) = p$$
$$P_{X_i}(X_i = 0) = 1-p$$

Find the mean function of the random process X s.t $\pi_i \circ X = X_i$

<u>Sol^n</u>. The mean function of process X, $m_X : \mathbb{N} \to \mathbb{R}$.

$$m_X(i) = \mathbb{E}[X_i] = 0\cdot(1-p) + 1\cdot p$$

<u>Exercise 2</u>: Let us define a random process S s.t.

$$\pi_n \circ S = S_n = S_0 + \sum_{i=1}^{n} X_i$$

. Let S_0 is a zero-mean RV. Find the mean function of process X.

<u>Sol^n</u>:

$$m_X(n) = \mathbb{E}[S_n]$$
$$= \mathbb{E}\left[S_0 + \sum_{i=1}^{n} X_i\right]$$
$$= \mathbb{E}[S_0] + \sum_{i=1}^{n} \mathbb{E}[X_i]$$
$$= np$$

<u>Exercise 3</u>. Let us consider the random process S as before. Let us define $\tau_n = \min\{n \in \mathbb{N} : S_n \geq n\}$. Let S' be a new process s.t.

Find $\mathbb{P}(\{\tau_n = n\})$.

<u>Sol^n</u>:

$$\{\tau_n = n\} = \{X_n = 1\} \cap \{S_{n-1} = n-1\}$$

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$$\mathbb{P}(\{X_n = 1\} \cap \{S_{n-1} = r-1\}) = \mathbb{P}(\{X_n = 1\}) \cdot \mathbb{P}(\{S_{n-1} = r-1\})$$

$$= \begin{cases} p \cdot {}^{n-1}C_{r-1} \cdot p^{r-1}(1-p)^{n-r} & \text{if } n \geq r \\ 0 & \text{otherwise} \end{cases}$$

<u>Exercise 4</u>. Define a random process X by

$\pi_t \circ X = X_t = A \cos(2\pi t + \theta)$ where $\Theta \sim \mathrm{Unif}(0, 2\pi)$.

<u>Sol^n</u>.

$$m_X(t) = \int_0^{2\pi} A \cos(2\pi t + \theta) \cdot \frac{1}{2\pi} \, d\theta$$

$$= \frac{A}{2\pi} \sin(2\pi t + \theta) \Big|_0^{2\pi}$$

$$= \frac{A}{2\pi} \left(\sin(2\pi t + 2\pi) - \sin(2\pi t) \right)$$

$$= 0$$

<u>Expectation of a random variable</u>

① <u>Simple RV</u> $X : \Omega \to \mathcal{X} \subset \mathbb{R}$, $|\mathcal{X}| < |\mathbb{N}|$

$$\mathbb{E}X = \sum_{x \in \mathcal{X}} x P_X(x)$$

② <u>MCT</u> Let $X : \Omega \to \mathbb{R}_+$. If $X_n \uparrow X$ then $\lim_{n \to \infty} \mathbb{E}X_n = \mathbb{E}[\lim_{n \to \infty} X_n]$

③ Any RV, X can be written as limit of simple RVs.

④ $$\mathbb{E}[X] = \lim_{n \to \infty} \mathbb{E}[X_n] = \int_{\mathbb{R}_+} x \, dF_X(x) = \begin{cases} \sum_{x \in \mathcal{X}} x \, p_X(x) & \text{– discrete RV} \\ \int x f_X(x) \, dx & \text{– Continuous RV.} \end{cases}$$

⑤ $X = X_+ - X_-$, $X_+ = \max\{X, 0\}$, $X_- = \min\{0, -X\}$

⑥ $\mathbb{E}X = \mathbb{E}X_+ - \mathbb{E}X_-$

Exercise 5. Show that Cauchy random variable has undefined mean.

Sol^n $X : \Omega \to \mathbb{R}$ is a Cauchy RV if the density of X.

$$f_X(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}$$

$X_+ : \Omega \to \mathbb{R}_+$ has the density f_{X_+} s.t.

$$f_{X_+}(x) = \begin{cases} \frac{1}{\pi} \frac{1}{1+x^2} & x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$\therefore \quad \mathbb{E}[X_+] = \int_{0+}^{\infty} x f_{X_+}(x)\, dx$$

$$= \frac{1}{\pi} \int_0^{\infty} \frac{x}{1+x^2}\, dx = \frac{1}{2\pi} \log(1+x^2) \Big|_0^{\infty}$$

$$= \infty$$

Similarly $\mathbb{E}[X_-] = \infty$.

Hence, $\mathbb{E}[X] = \mathbb{E}[X_+] - \mathbb{E}[X_-]$

$$= \infty - \infty$$