Tutorial 5

Monday, 5 September 2022 11:30 AM

Exercise 1: Let X and Y take values on $\{1,2\}$ and $\{-1,1\}$ respectively with joint mass function p_{XY} s.t.

$$p_{XY}(1,-1) = \frac{1}{5}, \quad p_{XY}(1,1) = \frac{1}{3}$$
$$p_{XY}(2,-1) = \frac{1}{5}, \quad p_{XY}(2,1) = \frac{4}{15}$$

Find p_X, p_Y, $\mathbb{E}X$, $\mathbb{E}Y$, $\mathbb{E}X^2$, $\mathbb{E}Y^2$, $\mathbb{E}XY$, $\operatorname{Var}X$, $\operatorname{Var}Y$, $\operatorname{cov}(X,Y)$, $\rho(X,Y)$.

Sol^n.

$$p_X(1) = p_{XY}(1,-1) + p_{XY}(1,1) = \frac{1}{5} + \frac{1}{3} = \frac{8}{15}$$
$$p_X(2) = p_{XY}(2,-1) + p_{XY}(2,1) = \frac{1}{5} + \frac{4}{15} = \frac{7}{15}$$
$$p_Y(-1) = p_{XY}(1,-1) + p_{XY}(2,-1) = \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$$
$$p_Y(1) = p_{XY}(1,1) + p_{XY}(2,1) = \frac{1}{3} + \frac{4}{15} = \frac{9}{15} = \frac{3}{5}$$

$$\mathbb{E}X = 1\cdot\frac{8}{15} + 2\cdot\frac{7}{15} = \frac{22}{15} \qquad \mathbb{E}Y = -1\cdot\frac{2}{5} + 1\cdot\frac{3}{5} = \frac{1}{5}$$
$$\mathbb{E}X^2 = 1^2\cdot\frac{8}{15} + 2^2\cdot\frac{7}{15} = \frac{36}{15} \qquad \mathbb{E}Y^2 = (-1)^2\cdot\frac{2}{5} + 1^2\cdot\frac{3}{5} = 1$$

$$\mathbb{E}XY = \sum_{x,y} x\cdot y\cdot p_{XY}(x,y) = 1\cdot(-1)\cdot\frac{1}{5} + 1\cdot1\cdot\frac{1}{3} + 2\cdot -1\cdot\frac{1}{5} + 2\cdot1\cdot\frac{4}{15}$$
$$= -\frac{1}{5} + \frac{1}{3} - \frac{2}{5} + \frac{8}{15} = \frac{4}{15}$$

$$\operatorname{Var}(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{36}{15} - \left(\frac{22}{15}\right)^2 = \frac{56}{225}$$
$$\operatorname{Var}(Y) = \mathbb{E}Y^2 - (\mathbb{E}Y)^2 = 1 - \left(\frac{1}{5}\right)^2 = \frac{24}{25}$$
$$\operatorname{Cov}(X,Y) = \mathbb{E}(X-\mathbb{E}X)(Y-\mathbb{E}Y)$$
$$= \sum_{x,y}(x-m_x)(y-m_y)\cdot p_{XY}(x,y)$$
$$= \sum_{x,y} xy\, p_{XY} - m_x m_y$$
$$= \frac{4}{15} - \frac{22}{15}\cdot\frac{1}{5} = -\frac{2}{75}$$

$$\rho(X,Y) = \frac{\operatorname{cov}(X,Y)}{\sqrt{\operatorname{Var}(X)\operatorname{Var}(Y)}} = \frac{-2/75}{\frac{\sqrt{56\times24}}{15\times5}} = -\frac{2}{8\sqrt{21}} = -\frac{1}{4\sqrt{21}}$$

- Uncorrelated RVs and Independent RVs.
  - Independence $\Rightarrow$ Uncorrelatedness
  - Uncorrelatedness $\not\Rightarrow$ Independence.

    Example - X and X^2, $X \sim Unif(-1,1)$

Exercise 2. Suppose that a RV X satisfies $\mathbb{E}X = 0, \mathbb{E}X^2 = 1, \mathbb{E}X^3 = 0$ and $\mathbb{E}X^4 = 1$ and let $Y = a + bX + cX^2$. Find ρ_{XY}.

Solⁿ: $Cov(X,Y) = \mathbb{E}XY - \mathbb{E}X\,\mathbb{E}Y$

$$= \mathbb{E}\left[aX + bX^2 + cX^3\right] = b$$

$$Var\,X = \mathbb{E}X^2 - (\mathbb{E}X)^2 = 1$$

$$Var\,Y = \mathbb{E}Y^2 - (\mathbb{E}Y)^2 = \mathbb{E}\left[a^2 + b^2X^2 + c^2X^4 + 2abX + 2acX^2 + 2bcX^3\right]$$

$$= a^2 + b^2 + c^2 + 2ac - (a+c)^2$$

$$= b^2$$

$$\therefore \rho_{XY} = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}} = \frac{b}{\sqrt{b^2}} = \frac{b}{|b|} = sign(b)$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, RV $X : \Omega \to \mathbb{R}$.

- pth moment of RV X : $\mathbb{E}X^k$
- pth absolute moment of RV X : $\mathbb{E}|X|^k$
- L^p Space : Collection of RVs s.t. pth absolute moment is bounded.

$$L^p = \{ X : \Omega \to \mathbb{R},\ X \text{ is RV} : (\mathbb{E}|X|^p)^{1/p} < \infty \}$$

Exercise 3. Show that L^2 is a vector space over $\mathbb{R}$.

Solⁿ: Let $X, Y \in L^2$, which means $\mathbb{E}|X|^2 < \infty$, $\mathbb{E}|Y|^2 < \infty$

We need to show that

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$$\text{A. } \mathbb{E}|X+Y|^2 < \infty \quad \text{and} \quad \text{B. } \mathbb{E}|\alpha X|^2 < \infty$$

$$\text{A.} \quad \mathbb{E}|X+Y|^2 = \mathbb{E}\left[(|X|+|Y|)^2\right]$$

$$= \mathbb{E}\left[|X|^2 + 2|X||Y| + |Y|^2\right] \qquad |x||y| = |xy|$$

$$\leq \mathbb{E}|X|^2 + 2\sqrt{\mathbb{E}X^2\,\mathbb{E}Y^2} + \mathbb{E}|Y|^2$$

$$< \infty$$

$$\text{B.} \quad \mathbb{E}|\alpha X|^2 = |\alpha|^2\,\mathbb{E}X^2 < \infty$$

Now check for associativity, commutativity of addition, additive identity, additive inverse, associativity of scalar multiplication distributivity of scalar multiplication with vector and field addition, and multiplicative identity in the field.

* Additive closure for L^p space : See Minkowski's inequality or,

$$\mathbb{E}|X+Y|^p \leq \mathbb{E}(|X|+|Y|)^p$$

$$\leq 2^p\,\mathbb{E}\max\{|x|,|Y|\}^p$$

$$= 2^p\,\mathbb{E}\max\{|X|^p,|Y|^p\}$$

$$\leq 2^p\,\mathbb{E}(|X|^p+|Y|^p) < \infty$$

<u>Markov Inequality</u>: X be non-negative RV. Then

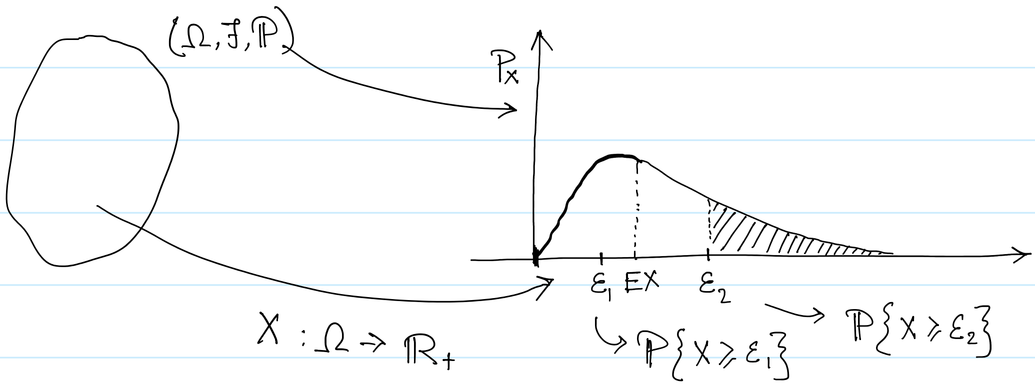
$$\mathbb{P}\{X \geq \varepsilon\} \leq \frac{\mathbb{E}X}{\varepsilon} \quad \text{for all } \varepsilon > 0$$

$$\mathbb{E}X = \int_\Omega X\,d\mathbb{P} = \int_\Omega X \mathbb{1}_{\{X<\varepsilon\}}\,d\mathbb{P} + \int_\Omega X \mathbb{1}_{\{X\geq\varepsilon\}}\,d\mathbb{P}$$

$$\geq \varepsilon \int \mathbb{1}_{\{X\geq\varepsilon\}}\,d\mathbb{P}$$

$$= \varepsilon\,\mathbb{P}\{X \geq \varepsilon\}$$

$$\Rightarrow \mathbb{P}\{X \geq \varepsilon\} \leq \frac{\mathbb{E}X}{\varepsilon}$$



Chebyshev Inequality.

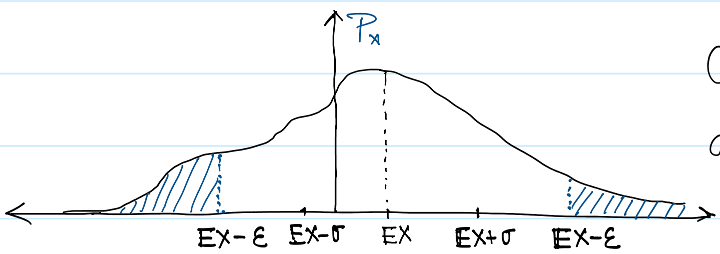
If X is a RV, $(X - \mathbb{E}X)^2$ is a RV too.
Moreover it is a +ve RV.

$$\therefore \quad \mathbb{P}\{(X - \mathbb{E}X)^2 \geq \varepsilon^2\} \leq \frac{1}{\varepsilon^2} \cdot \mathbb{E}[(X - \mathbb{E}X)^2]$$

$$= \frac{\sigma^2}{\varepsilon^2}$$

$$\Rightarrow \mathbb{P}\{|X - \mathbb{E}X| \geq \varepsilon\} \leq \left(\frac{\sigma}{\varepsilon}\right)^2$$

$$\Rightarrow \mathbb{P}\{X \geq \mathbb{E}X + \varepsilon \text{ or } X \leq \mathbb{E}X - \varepsilon\} \leq \left(\frac{\sigma}{\varepsilon}\right)^2$$



Chebyshev's inequality gives an upper bound of the shaded area.

Exercise 4: A biased coin, with $\Pr(H) = \frac{1}{10}$, is flipped 200 times consequitively. Give an upperbound on the probability that it lands heads at least 120 times.

Sol^n Let X_i be the random variable corresponding to i^{th} toss.

$$X_i = 1 \quad \text{if outcome is head, else } X_i = 0.$$

Let $S = \sum_{i=1}^{200} X_i$ be the random variable that takes the Number of total heads.

$i=1$

Number of total heads.

$$\mathbb{E}S = \mathbb{E}\sum_{i=1}^{200} X_i = \sum_{i=1}^{200} \mathbb{E}X_i = 20$$

$$\therefore \; P\{S \geqslant 120\} \leqslant \frac{20}{120} = \frac{1}{6}$$

Exercise 5 : An unbiased coin is tossed 100 times. Give a lower bound on the probability that number of heads is greater than equal to 35 and less than equals to 65.

Sol^n: $S = \sum_{i=1}^{100} X_i \qquad \mathbb{E}S = \sum_{i=1}^{100} \mathbb{E}X_i = 100 \cdot \frac{1}{2} = 50.$

$$Var(S) = \mathbb{E}\left[\left(\sum_{i=1}^{100} X_i - \mathbb{E}X_i\right)^2\right] = 100 \; \mathbb{E}\left[(X_1 - \mathbb{E}X_1)^2\right] = 25$$

$$\begin{aligned}
P(\{S \geqslant 35\} \cup \{S \leqslant 65\}) &= P(|S - \mathbb{E}S| \leqslant 15) \\
&= 1 - P(|S - \mathbb{E}S| > 15) \\
&\leqslant 1 - P(|S - \mathbb{E}S| \geqslant 15) \\
&\leqslant 1 - \frac{Var(S)}{15^2} \\
&= 1 - \frac{1}{9} = \frac{8}{9}
\end{aligned}$$

Exercise 6 : An unbiased coin is tossed 100 times. Give an upper bound on the probability that number of heads $S \geqslant 55$

Sol^n. $S = \sum_{i=1}^{100} X_i \qquad \mathbb{E}S = \sum_{i=1}^{100} \mathbb{E}X_i = 100 \cdot \frac{1}{2} = 50.$

Using Markov inequality,

$$P\{S \geqslant 55\} \leqslant \frac{\mathbb{E}S}{55} = \frac{50}{55} = 0.909$$

Let us now use Chernoff bound.

$$\mathbb{E}[e^{\theta S}] = \sum_{S=1}^{100} e^{\theta s} p_S(s) = \sum_{S=1}^{100} e^{\theta s} \, {}^{100}C_s \, p^s (1-p)^{100-s}$$

$$E[e^{\theta S}] = \sum_{S=1}^{100} e^{\theta s} p_S(s) = \sum_{S=1}^{100} e^{\theta s} \, {}^{100}C_s \, p^s (1-p)^{100-s}$$

$$= \left(1 - p + p e^{\theta s}\right)^{100}$$

$$= \frac{1}{2^{100}} \left(1 + e^{\theta s}\right)^{100}$$

$$P(S \geqslant 55) \leqslant \frac{e^{-\theta \cdot 55}}{2^{100}} \left(1 + e^{\theta s}\right)^{100} = f(\theta)$$

We can minimize $f(\theta)$ on θ which will give the lowest upper bound.

$$f'(\theta) = -55 e^{-55\theta} \left(1 + e^{\theta}\right)^{100} \frac{1}{2^{100}} + e^{-55\theta} \cdot 100 \left(1 + e^{\theta}\right)^{99} \cdot e^{\theta} \cdot \frac{1}{2^{100}}$$

$f'(\theta^*) = 0$ yields,

$$e^{-\theta^*} = \frac{9}{11} \Rightarrow \theta^* = \ln \frac{11}{9}$$

Therefore, $f(\theta^*) = \left(\frac{11}{9}\right)^{-55} \cdot \left(1 + \frac{11}{9}\right)^{100} \cdot \frac{1}{2^{100}} = 0.606$