Tutorial 6

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Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let X be an RV on $(\Omega, \mathcal{F}, \mathbb{P})$.

Charateristics function : $\phi_X(u) \triangleq \mathbb{E} e^{juX}$

Example 1. Let $X \sim Exp(\lambda)$. Find $\phi_X(u)$.

- $f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$

$$\therefore \phi_X(u) = \mathbb{E} e^{juX} = \int_{\mathbb{R}} e^{jux} \cdot \lambda e^{-\lambda x} dx$$

$$= \int_0^\infty \lambda e^{x(-\lambda + ju)} dx$$

$$= \frac{\lambda}{-\lambda + ju} e^{x(-\lambda + ju)} \Big|_0^\infty$$

$$= \frac{\lambda}{\lambda - ju}$$

Example 2. Let $X \sim Pois(\lambda)$. Find $\phi_X(u)$.

- $P_X(k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}, \quad k \in \mathbb{Z}_+$

$$\therefore \mathbb{E} e^{juX} = \sum_{k=0}^{\infty} e^{juk} e^{-\lambda} \cdot \frac{\lambda^k}{k!} = \sum_{k=0}^{\infty} e^{-\lambda} \cdot \frac{(\lambda e^{ju})^k}{k!}$$

$$= e^{-\lambda} \cdot e^{\lambda e^{ju}}$$

$$= e^{\lambda(e^{ju} - 1)}$$

Exercise 1. $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$, $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ are uncorrelated. Show that $X_1 + X_2 \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

Sol^n

We know that if two Gaussian random variables are uncorrelated, they are independent too.

We will show that if X_1, X_2 are independent,

$$\phi_{X_1+X_2}(u) = \phi_{X_1}(u) \cdot \phi_{X_2}(u)$$

If X_1, X_2 are independent, then e^{juX_1} and e^{juX_2} are independent. Hence e^{juX_1}, e^{juX_2} are uncorrelated, i.e.

$$\mathbb{E}\left[e^{juX_1} \cdot e^{juX_2}\right] = \mathbb{E}\left[e^{juX_1}\right] \mathbb{E}\left[e^{juX_1}\right]$$

$$\Rightarrow \phi_{X_1+X_2}(u) = \mathbb{E}\left[e^{ju(X_1+X_2)}\right] = \phi_{X_1}(u) \cdot \phi_{X_2}(u)$$

Now, as $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$, $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$,

$$\phi_{X_1+X_2}(u) = \exp\left(-\frac{u^2\sigma_1^2}{2} + ju\mu_1\right) \exp\left(-\frac{u^2\sigma_2^2}{2} + ju\mu_2\right)$$

$$= \exp\left(-\frac{u^2}{2}(\sigma_1^2 + \sigma_2^2) + ju(\mu_1 + \mu_2)\right)$$

$$\Rightarrow X_1 + X_2 \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2) \qquad \square$$

Probability Generating Function : $\Psi_X(z) = \sum_{x \in \mathcal{X}} z^x P_X(x)$

where $\mathcal{X} \subseteq \mathbb{Z}_+$

Example 3. Let $X \sim \text{Binomial}(n, p)$. Find $\Psi_X(z)$

$$P_X(k) = \begin{cases} {}^nC_k \, p^k (1-p)^{n-k} & k = 0, 1, \ldots, n \\ 0 & \text{otherwise.} \end{cases}$$

$$\therefore \Psi_X(z) = \sum_{k=0}^{n} {}^nC_k (pz)^k (1-p)^{n-k}$$

$$= (1 - p + pz)^n$$

Conditional Expectation

* $\mathbb{E}[X|Y]$ is a random variable.

Example : Let X and Y take values on $\{1, 2\}$ and $\{-1, 1\}$

<u>Example</u>: Let X and Y take values on $\{1,2\}$ and $\{-1,1\}$ respectively with joint mass function P_{XY} s.t.

$$P_{XY}(1,-1) = \frac{1}{5}, \quad P_{XY}(1,1) = \frac{1}{3}$$

$$P_{XY}(2,-1) = \frac{1}{5}, \quad P_{XY}(2,1) = \frac{4}{15}$$

Find $\mathbb{E}[X|Y]$ and $\mathbb{E}[Y|X]$ and there distributions.

<u>Sol^n</u>.

$$P_{X|Y}(x=1 \mid y=-1) = \frac{P_{XY}(1,-1)}{P_Y(-1)} = \frac{1}{2}$$

$$P_{X|Y}(x=2 \mid y=-1) = \frac{P_{XY}(2,-1)}{P_Y(-1)} = \frac{1}{2}$$

$$P_{X|Y}(x=1 \mid y=1) = \frac{P_{XY}(1,1)}{P_Y(1)} = \frac{\frac{1}{3}}{\frac{3}{5}} = \frac{5}{9}$$

$$P_{X|Y}(x=2 \mid y=1) = \frac{P_{XY}(2,1)}{P_Y(1)} = \frac{4}{9}$$

$$\mathbb{E}[X \mid Y=-1] = 1 \cdot P_{X|Y}(x=1 \mid y=-1) + 2\, P_{X|Y}(x=2 \mid y=-1)$$
$$= \frac{3}{2}$$

$$\mathbb{E}[X \mid Y=1] = 1 \cdot P_{X|Y}(x=1 \mid y=1) + 2 \cdot P_{X|Y}(x=2 \mid y=1)$$
$$= 1 \cdot \frac{5}{9} + 2 \cdot \frac{4}{9} = \frac{13}{9}$$

look at the set

$$\left\{\omega : \mathbb{E}[X(\omega) \mid Y(\omega)] = \frac{3}{2}\right\} = \left\{\omega : Y(\omega) = -1\right\}$$

$$\Rightarrow P\left\{\omega : \mathbb{E}[X(\omega) \mid Y(\omega)] = \frac{3}{2}\right\} = P\left\{\omega : Y(\omega) = -1\right\}$$
$$= P(Y=-1) = \frac{2}{5}$$

Similarly, $P\left\{\omega : \mathbb{E}[X(\omega) \mid Y(\omega)] = \frac{13}{9}\right\} = \frac{3}{5}$

$$\therefore \quad \mathbb{E}[X|Y] = \begin{cases} \frac{3}{2} & \text{w.p. } \frac{2}{5} \\ \frac{13}{9} & \text{w.p. } \frac{3}{5} \end{cases}$$

<u>Verify</u> $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$

$$\mathbb{E}\big[\mathbb{E}[X|Y]\big] = \frac{3}{2}\cdot\frac{2}{5} + \frac{13}{9}\cdot\frac{3}{5} = \frac{22}{15} = \mathbb{E}[X]$$

· <u>Law of iterated Expectation</u> :

* We know $\mathbb{E}\big[\mathbb{E}[X|\mathcal{G}]\big] = \mathbb{E}X$.
* Also $\mathbb{E}[X|Y] := \mathbb{E}[X|\sigma(Y)]$

$\therefore$ $\mathbb{E}\big[\mathbb{E}[X|Y]\big] = \mathbb{E}X$.

<u>Exercise 2</u>. $Z = \sum_{i=1}^{N} X_i$. X_i's are iid and $N : \Omega \to \mathbb{Z}_+$ is independent of X_i's. M_X and M_N are given. Find M_Z.

<u>Soln</u>.

$$\begin{aligned} M_Z(t) &= \mathbb{E}\left[e^{t\sum_{i=1}^{N} X_i}\right] \\ &= \mathbb{E}\left[\prod_{i=1}^{N} e^{tX_i}\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\prod_{i=1}^{N} e^{tX_i} \,\Big|\, N\right]\right] \\ &= \mathbb{E}\left[\prod_{i=1}^{N} \mathbb{E}[e^{tX_i}]\right] \\ &= \mathbb{E}\left[M_X(t)^N\right] \\ &= \mathbb{E}\left[e^{N \log M_X(t)}\right] \\ &= M_N(\log M_X(t)) \end{aligned}$$

or, $M_Z(t) = \mathbb{E}\left[M_X(t)^N\right] = \Psi_N\left[M_X(t)\right]$

<u>Exercise 3</u>. Take $X_i \sim \text{Exp}(\mu)$, $N \sim \text{Geom}(p)$ and $Z = \sum_{i=1}^{N} X_i$. Find distribution of Z given X_i's are independent.

<u>Soln</u>. $M_X(t) = \frac{\mu}{\mu - s}$, $s < \mu$

$$\Psi_N(z) = \sum^{\infty} z^k (1-p)^{k-1} \cdot p = \frac{pz}{1 - z(1-p)}, \quad |z| < \frac{1}{1-p}$$

$\mu - s$

$$\Psi_N(z) = \sum_{k=0}^{\infty} z^k (1-p)^{k-1} \cdot p = \frac{pz}{1 - z(1-p)} \quad , \quad |z| < \frac{1}{1-p}$$

$$\therefore \quad M_Z(t) = \Psi_N\left(M_X(t)\right)$$

$$= \frac{p\left(\frac{\mu}{\mu - s}\right)}{1 - \left(\frac{\mu}{\mu - s}\right)(1-p)} \quad , \quad \left|\frac{\mu}{\mu - s}\right| < \frac{1}{1-p}$$

$$\left|\frac{\mu}{\mu - s}\right| = \frac{\mu}{\mu - s} < \frac{1}{1-p} \quad \Rightarrow \quad \mu - \mu p < \mu - s$$

$$\Rightarrow \quad s < \mu p$$

$$M_Z(t) = \frac{\mu p}{\mu p - s} \quad , \quad s < \mu p$$

$$\Rightarrow \quad Z \sim \mathrm{Exp}(\mu p)$$