Tutorial 7

Monday, 19 September 2022 10:21 AM

Convergence of Sequence of Numbers:

Let (a_n) be a sequence of real numbers. We say (a_n) converges to (a) if for every $\varepsilon > 0$ $\exists N_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq N_\varepsilon$ $|a_n - a| \leq \varepsilon$.

Limit superior $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup_{m \geq n} a_n$

Limit inferior $\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf_{m \geq n} a_n$

* One can show that $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = a$ iff $\lim_{n\to\infty} a_n = a$

Convergence of functions:

Let $(f_n : X \to Y)$ be a sequence of functions. We say (f_n) converges to $f : X \to Y$ everywhere if $\forall x \in X$ $(f_n(x)) \subset Y$ converges to $f(x) \in Y$.

Convergence of Random Variables:

Random variables are functions, hence we can talk about convergence everywhere of random variables.

Let $(X_n : \Omega \to \mathbb{R})$ be a sequence of functions. We say (X_n) converges to $X : \Omega \to \mathbb{R}$ everywhere if $\forall \omega \in \Omega$ $(X_n(\omega)) \subset \mathbb{R}$ converges to $X(\omega)$.

* For a random variable we are hardly interested in convergence of (X_n) to X for all $\omega \in \Omega$. Rather we look at the collection of ω's with probability measure 1.

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* If $P(\{\omega \mid X_n(\omega) \to X(\omega)\}) = 1$ then we say $X_n \xrightarrow{a.s.} X$.

– Check that this is same as the definition given in the lecture notes $\big(P(\text{exception set}) = 0\big)$.

* We can define several other relations between random variables in almost sure sense, e.g., monotonicity, equality.

<u>Example-1</u>. (Convergence almost surely but not pointwise)

Let $(\Omega, \mathcal{F}, \lambda) = ([0,1], \mathcal{B}([0,1]), \lambda)$ where λ is the Lebesgue measure. Let us consider a sequence (X_n) s.t.

$$X_n(\omega) = \begin{cases} n & \text{if } \omega \in [0, \frac{1}{n}] \\ 0 & \text{otherwise} \end{cases}$$

Therefore
$$\lambda(\{\omega \mid X_n(\omega) = n\}) = \frac{1}{n}$$
$$\lambda(\{\omega \mid X_n(\omega) = 0\}) = 1 - \frac{1}{n}$$

When $\omega \neq 0$, $\lim_{n\to\infty} X_n(\omega) = 0$, but $(X_n(0))$ diverges. So, $X_n \not\to 0$ everywhere. So the exception set $N = \{\omega \mid \omega = 0\}$ and $\lambda(N) = 0$.

Hence $X_n \xrightarrow{a.s.} 0$

* If $\lim_{n\to\infty} P(\{\omega \mid |X_n(\omega) - X(\omega)| > \varepsilon\}) = 0$ for every $\varepsilon > 0$ then we say $X_n \xrightarrow{i.p.} X$.

It's trivial to show that $X_n \xrightarrow{i.p.} 0$ in example 1.

Question : a. Does $X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{i.p.} X$? Yes

b. Does $X_n \xrightarrow{i.p.} X \Rightarrow X_n \xrightarrow{a.s.} X$? In general, No.

a. $A_n(\varepsilon) = \{\omega \mid |X_n(\omega) - X(\omega)| > \varepsilon\}$. Fix $\varepsilon > 0$.

If for a fixed $\omega_0 \in \Omega$, $X_n(\omega_0) \to X(\omega_0)$, i.e., $\omega_0 \in N^c$,

then $\exists\, N_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq N_\varepsilon$, $|X_n(\omega_0) - X(\omega_0)| \leq \varepsilon$.

In other words, $\forall n \geq N_\varepsilon$, $\omega_0 \notin A_n(\varepsilon)$.

$$\Rightarrow N^c \cap \lim_{n\to\infty} A_n(\varepsilon) = \phi$$

$$\Rightarrow P(N^c) + P\left(\lim_{n\to\infty} A_n(\varepsilon)\right) \leq 1$$

$$\Rightarrow P\left(\lim_{n\to\infty} A_n(\varepsilon)\right) = 0$$

$$\Rightarrow \lim_{n\to\infty} P(A_n(\varepsilon)) = 0$$

Infinitely often and all but finitely many :

Let (A_n) be a sequence of events

$$\{A_n \text{ i.o.}\} = \left\{\omega\in\Omega : \sum_{i=1}^{\infty} \mathbb{1}_{A_i}(\omega) = \infty\right\}$$

$$= \{\omega \in \Omega : \text{Given any } N \in \mathbb{N}, \exists m \geq N \text{ s.t. } \omega \in A_m\}$$

$$\limsup_{n\to\infty} A_n = \bigcap_{N\in\mathbb{N}} \bigcup_{m\geq N} A_m = \bigcap_{N\in\mathbb{N}} \{\omega\in\Omega : \omega \in A_m \text{ for some } m \geq N\}$$

$$= \{\omega\in\Omega : \text{Given any } N\in\mathbb{N}\ \exists\, m \geq N \text{ s.t. } \omega \in A_m\}$$

$$\therefore \{A_n \text{ i.o.}\} = \limsup_{n\to\infty} A_n$$

$$\{A_n \text{ a.b.f.m}\} = \{\omega\in\Omega : \mathbb{1}_{A_m}(\omega) = 1 \ \forall\, m \geq n(\omega)\}$$

$$\liminf_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} A_m = \bigcup_{n\in\mathbb{N}} \{\omega\in\Omega : \omega \in A_m\ \forall\, m \geq n\}$$

$$= \{\omega \in \Omega : \omega \in A_m \ \forall\, m \geq n \text{ for some } n \in \mathbb{N}\}$$

$$\therefore \quad \{A_n \text{ a.b.f.m}\} = \liminf_{n\to\infty} A_m$$

Borel-Cantelli lemma : $\sum_{n\in\mathbb{N}} P(A_n) < \infty \Rightarrow P(A_n \text{ i.o.}) = 0$

proof:

$$P(A_n \text{ i.o.}) = P\left(\limsup_{n\to\infty} A_n\right)$$

$$= P\left(\bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} A_m\right)$$

$$= \lim_{n\to\infty} P\left(\bigcup_{m\geq n} A_m\right)$$

$$\leq \lim_{n\to\infty} \sum_{m\geq n} P(A_m)$$

$$= 0$$

Let (a_n) be a sequence in $\mathbb{R}_{\geq 0}$.

$$S_n = \sum_{i=1}^{n} a_n \,,\quad S_n \to S < \infty$$

$$\Rightarrow |S_{n-1} - S| \to 0$$

$$\Rightarrow \sum_{i=n}^{\infty} a_n \to 0$$

* Independence (A_n) is not required.

Lemma . Show that $P(A_n \text{ i.o.}) = 1$ iff $\sum_{n\in\mathbb{N}} P(A_n) = \infty$

Proof: Let $\sum_{n\in\mathbb{N}} P(A_n) = \infty$.

$$(A_n \text{ i.o.}) = \left\{\omega \in \Omega : \sum \mathbb{1}_{A_i}(\omega) = \infty\right\}$$

$$= \left\{\omega \in \Omega : \sum \mathbb{1}_{A_i}(\omega) < \infty\right\}^c$$

$$= \left\{\omega \in \Omega : \mathbb{1}_{A_n^c}(\omega) = 1 \ \forall\, n \geq n_0(\omega)\right\}^c$$

$$= \{A_n^c \text{ a.b.f.m}\}^c$$

$$\Rightarrow P(A_n \text{ i.o.}) = 1 - P(A_n^c \text{ a.b.f.m})$$

$$= 1 - P\left(\liminf_{n\to\infty} A_n^c\right)$$

$$= 1 - P\left(\bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} A_m^c\right)$$

$$= 1 - P\left(\bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} A_m^c\right)$$

$$= 1 - \lim_n \lim_m P\left(\bigcap_{k=n}^{m} A_k^c\right)$$

$$= 1 - \lim_n \lim_m \prod_{k=n}^{m} \left(1 - P(A_k)\right)$$

$$\geq 1 - \lim_n \lim_m \exp\left(-\sum_{k=n}^{m} P(A_k)\right) \qquad \text{as } 1-x \leq e^{-x}$$

$$= 1 - \lim_n \exp\left(-\sum_{k=n}^{\infty} P(A_k)\right)$$

$$= 1$$

$$\Rightarrow P(A_n \text{ i.o.}) = 1.$$

* Borel–Cantelli lemma is the contra-positive of the converse.

Example: Let us consider the random experiment of infinite coin tossing. H_n be the event of getting head on the n^{th} toss and $P(H_n) = \frac{1}{n}$. Then there will be infinite heads in total with probability 1.

Change $P(H_n) = \frac{1}{n^2}$. There will be finite heads with probability 1. Remember that tosses must be independent for this conclusion.

Theorem: If $\forall\, \varepsilon > 0$ $\sum_n P(A_n(\varepsilon)) < \infty$, then $X_n \xrightarrow{a.s.} X$.