Tutorial 8

Monday, 19 September 2022 8:58 PM

Recap:

1. Pointwise convergence: $X_n \xrightarrow{P.W} X$ if $\lim_{n\to\infty} X_n(\omega) = X(\omega)$ $\forall \omega \in \Omega$.
2. Almost sure convergence: $X_n \xrightarrow{a.s.} X$ if $P(\{\omega \in \Omega : X_n(\omega) \to X(\omega)\}) = 1$
3. Convergence in probability: $X_n \xrightarrow{i.P.} X$ if $\lim_{n\to\infty} P(A_n(\varepsilon)) = 0$

   where $A_n(\varepsilon) = \{\omega \in \Omega : |X_n(\omega) - X(\omega)| > \varepsilon\}$
4. $P.W \Rightarrow a.s. \Rightarrow i.p.$, $i.p. \not\Rightarrow a.s.$ in general.
5. $(A_n \text{ i.o.}) = \limsup_n A_n$ and $(A_n \text{ a.b.f.m}) = \liminf_n A_n$
6. Borel Cantelli Lemma: $\sum_{n\in\mathbb{N}} P(A_n) < \infty \Rightarrow P(A_n \text{ i.o.}) = 0$
7. Borel 0-1 law: If A_n's are indendent then

$$P(A_n \text{ i.o.}) = \begin{cases} 0 & \text{if } \sum_{n\in\mathbb{N}} P(A_n) < \infty \\ 1 & \text{if } \sum_{n\in\mathbb{N}} P(A_n) = \infty \end{cases}$$

8. If $\forall \varepsilon > 0$ $\sum_n P(A_n(\varepsilon)) < \infty$, then $X_n \xrightarrow{a.s} X$

Proof. For fixed $\varepsilon > 0$, $\because$ by Borel-Cantelli lemma

$$P(A_n(\varepsilon) \text{ i.o.}) = P(\limsup_n A_n(\varepsilon)) = 0$$

$$\Rightarrow P((\limsup_n A_n(\varepsilon))^c) = 1$$

$$\Rightarrow P(\liminf_n A_n(\varepsilon)^c) = 1$$

Let $\liminf_n A_n(\varepsilon)^c =: B_\varepsilon$

We know that

$$N^c = \{\omega \in \Omega : X_n(\omega) \to X(\omega)\}$$

$$= \{\omega \in \Omega : \text{for every } p \in \mathbb{N}\ \exists N \in \mathbb{N} \text{ s.t } \forall n \geq N\ |X_n(\omega) - X(\omega)| < \varepsilon\}$$

$$= \bigcap \{\omega \in \Omega : \omega \in (A_n(\tfrac{1}{p})^c \text{ a.b.f. m})\}$$

$$= \bigcap_{p\in\mathbb{N}} \left\{ \omega \in \Omega : \omega \in \left(A_n\left(\tfrac{1}{p}\right)\right)^c \text{ a.b.f. } m \right\}$$

$$= \bigcap_{p\in\mathbb{N}} B_{\frac{1}{p}}$$

$$\Rightarrow P(N^c) = P\left(\bigcap_{p\in\mathbb{N}} B_{1/p}\right) = 1 \qquad \square$$

Note: for every $\varepsilon > 0$, $\sum_{n\in\mathbb{N}} P(A_n(\varepsilon)) < \infty \Rightarrow \lim_{n\to\infty} A_n(\varepsilon) = 0$ for any $\varepsilon > 0$
$\Rightarrow X_n \xrightarrow{i.p.} X$.

Note: We showed that for every $\varepsilon > 0$, $P(A_n(\varepsilon) \text{ i.o.}) = 0 \Rightarrow X_n \xrightarrow{a.s.} X$
It is also possible to show that $X_n \xrightarrow{a.s} X \Rightarrow$ for every $\varepsilon > 0$ $P(A_n(\varepsilon) \text{ i.o.}) = 0$.

* Verify for the examples given in the lecture.

9. Convergence in $\mathcal{L}^p$: $X_n \xrightarrow{\mathcal{L}^p} X$ if $\lim_{n\to\infty} \mathbb{E}|X_n - X|^p = 0$

For the particular case of $p = 2$. We say that $X_n \xrightarrow{m.s.} X$.

10. $\mathcal{L}^p \Rightarrow$ i.p.

11. $\mathcal{L}^p \not\Rightarrow$ a.s.

12. L^2 weak law of large numbers: X_n's are uncorrelated. $\mathbb{E}X_n = \mu$, $Var(X_n) = \sigma^2$ $\forall n \in \mathbb{N}$. $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_n$. Then $\bar{X}_n \xrightarrow{\mathcal{L}^p,\ i.p.} \mu$

Note: L^2 as $Var(X_n) = \sigma^2$, Weak $\Rightarrow$ not almost sure.

13. L^1 Weak law of large numbers: X_n's are i.i.d, $\mathbb{E}X_n = \mu$ $\forall n \in \mathbb{N}$, then $\bar{X}_n \xrightarrow{i.p.} \mu$.

14. Monotone Convergence Theorem: X_n's are non-decreasing, L^1. If $\lim_{n\to\infty} X_n(\omega) = X(\omega)$ $\forall \omega \in \Omega$ then $\lim_{n\to\infty} \mathbb{E}X_n = \mathbb{E}X$.

15. Dominated Convergence Theorem: X_n's are in L^1, $|X_n| \le Y \in L^1$ and $\lim_{n\to\infty} X_n(\omega) = X(\omega)$ $\forall \omega \in \Omega$ then $\lim_{n\to\infty} \mathbb{E}X_n = \mathbb{E}X$.

16. Convergence in distribution: $X_n \xrightarrow{D} X$ if $\lim F_{X_n}(x) = F_X(x)$ $\forall x \in \mathbb{R}$

16. Convergence in distribution : $X_n \xrightarrow{D} X$ if $\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \ \forall x \in \mathbb{R}$

17. $X_n \xrightarrow{D} X \iff \lim_{n\to\infty} \phi_{X_n}(u) = \phi_X(u)$ for every $u \in \mathbb{R}$.

<u>Exercise 1</u> Suppose that $((X_1, Y_1), (X_2, Y_2), \dots)$ is a sequence of random vectors s.t. $P(\{X_n \geq Y_n\}) = \alpha^n$ for some $\alpha \in (0,1)$. Show that $P(\{X_n \geq Y_n\} \text{ i.o.}) = 0$.

<u>Sol^n</u>. $\sum_{n \in \mathbb{N}} P(\{X_n \geq Y_n\}) = \frac{1}{1-\alpha} < \infty$. Hence

$$P(\{X_n \geq Y_n\} \text{ i.o}) = 0.$$

<u>Exercise 2</u> : $(X_n : n \in \mathbb{N})$ is a sequence of independent random variables with marginal pmf s given by

$$P\left(X_n = \tfrac{1}{2}\left(1 - \tfrac{1}{n}\right)\right) = P\left(X_n = \tfrac{1}{2}\left(1 - \tfrac{1}{n}\right)\right) = \tfrac{1}{2}$$

a. Show that the sequence converge in almost sure sense and find the limit.

b. Does X_n converge in $\mathcal{L}^2$?

<u>Sol^n</u>. a. For fixed $p \in \mathbb{N}$,

$$P(A_n(\tfrac{1}{p})) = P\left(\left\{\omega \in \Omega \,\middle|\, |X_n(\omega) - \tfrac{1}{2}| > \tfrac{1}{p}\right\}\right)$$

$$= P\left(\left\{\omega \in \Omega \,\middle|\, \tfrac{1}{2} - \tfrac{1}{p} \leq X_n(\omega) \leq \tfrac{1}{2} + \tfrac{1}{p}\right\}^c\right)$$

$$= P\left(\left\{\omega \in \Omega \,\middle|\, \tfrac{1}{2}\left(1 - \tfrac{2}{p}\right) \leq X_n(\omega) \leq \tfrac{1}{2}\left(1 - \tfrac{2}{p}\right)\right\}^c\right)$$

$$= \begin{cases} 0 & \text{if } n \geq \frac{p}{2} \\ 1 & \text{if } n < \frac{p}{2} \end{cases}$$

We have two observations

1. $\lim_{n\to\infty} P(A_n(\tfrac{1}{p})) = 0$ for every $p \in \mathbb{N}$

2. $\lim \sum^{n} P(A_i(\tfrac{1}{p})) < \frac{p}{2} < \infty$ for every $p \in \mathbb{N}$

1. $\lim_{n\to\infty} P(A_n(p)) = 0$ for every $p \in \mathbb{N}$

2. $\lim_{n\to\infty} \sum_{i=1}^{n} P\left(A_i\left(\frac{1}{p}\right)\right) < \frac{p}{2} < \infty$ for every $p \in \mathbb{N}$

$\Rightarrow \quad X_n \xrightarrow{a.s.} \frac{1}{2}$.

b. $$\mathbb{E}\left[\left(X_n - \tfrac{1}{2}\right)^2\right] = \left(\tfrac{1}{2}\left(1-\tfrac{1}{n}\right) - \tfrac{1}{2}\right)^2 \cdot \tfrac{1}{2} + \left(\tfrac{1}{2}\left(1+\tfrac{1}{n}\right) - \tfrac{1}{2}\right)^2$$
$$= \frac{1}{4n^2}$$

$$\lim_{n\to\infty} \mathbb{E}\left[\left(X_n - \tfrac{1}{2}\right)^2\right] = \lim_{n\to\infty} \frac{1}{4n^2} = 0$$

Hence, $X_n \xrightarrow{\mathcal{L}^2} 0$

<u>Exercise 3</u>: Let $U \sim \text{Unif}([0,1])$ and let $X_n = \frac{(-1)^n U}{n}$ for $n \geq 1$.

a. Show that X_n converges in almost sure sense. Find the limit.

b. Show that X_n converges in mean-squared sense. Find the limit.

<u>Sol^n</u>: a. Claim: $X_n \xrightarrow{a.s.} 0$

Fix $\varepsilon > 0$ arbitrarily.
Let $A_n(\varepsilon) = \{\omega \in \Omega \mid |X_n(\omega)| > \varepsilon\}$. If $P(A_n(\varepsilon) \text{ i.o.}) = 0$ then, $X_n \xrightarrow{a.s.} 0$.

We will show that $P\left(\liminf_n A_n(\varepsilon)^c\right) = 1$.

$$\liminf_n A_n(\varepsilon)^c = \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} \{\omega \in \Omega \mid |X_m(\omega)| \leq \varepsilon\}$$
$$= \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} \{\omega \in \Omega \mid U(\omega) \leq m\varepsilon\}$$
$$= \bigcup_{n\in\mathbb{N}} \{\omega \in \Omega \mid U(\omega) \leq n\varepsilon\}$$
$$= \{\omega \in \Omega \mid U(\omega) \in \mathbb{R}\}$$
$$= \Omega$$

$$\Rightarrow \quad P\left(\liminf_n A_n(\varepsilon)^c\right) = 1$$

It is to see that, $\liminf_n A_n^c = \left(\limsup A_n\right)^c$

$$\therefore \quad P\left(\limsup_n A_n(\varepsilon)\right) = P\left(A_n(\varepsilon) \text{ i.o.}\right) = 0$$

$$\Rightarrow \quad X_n \xrightarrow{a.s} X.$$

b. $\mathbb{E}\left[|X_n|^2\right] = \frac{1}{n^2}\mathbb{E}\left[U^2\right] = \frac{1}{3n^2}$

$$\Rightarrow \lim_{n\to\infty} \mathbb{E}\left[|X_n|^2\right] = 0$$

$$\therefore \quad X_n \xrightarrow{m.s.} 0.$$

<u>Exercise 4</u> : Let $\Theta \sim \text{Unif}([0,2\pi])$. Show that $X_n = \cos(n\Theta)$ does not converge in almost sure sense.

<u>Sol^n</u>: Define $N^c = \{\omega \in \Omega \mid \cos(n\Theta(\omega)) \text{ converges}\}$.

Let for some $\theta \in [0,2\pi]$, $\cos(n\theta)$ converges to $c \in [-1,1]$

We know that

$$\cos((n+1)\theta) + \cos((n-1)\theta) = 2\cos(n\theta)\cos\theta$$

Taking limit $n \to \infty$,

$$c + c = 2c\cos\theta$$

$$\Rightarrow \cos\theta = 1 \quad \Rightarrow \theta = 0, 2\pi.$$

$$\therefore \quad N^c = \{\omega \in \Omega \mid \Theta(\omega) = 0 \text{ or } \Theta(\omega) = 2\pi\}$$

$\Rightarrow P(N^c) = 0$, hence (X_n) does not converge in almost sure sense.

<u>Exercise 5</u> : Let $\Theta \sim \text{Unif}([0,2\pi])$. Show that $\left|1 - \frac{\Theta}{\pi}\right|^n$ converges to 0 in distribution.

<u>Sol^n</u>.

$$P\left(\left|\left|1-\frac{\theta}{\pi}\right|^n - 0\right| > \varepsilon\right) = 1 - P\left(\left|1-\frac{\theta}{\pi}\right| \leq \varepsilon^{1/n}\right)$$

$$= 1 - P\left(1-\varepsilon^{1/n} < \frac{\theta}{\pi} < 1+\varepsilon^{1/n}\right)$$

$$= 1 - P\left(\pi\left(1-\varepsilon^{1/n}\right) < \theta < \pi\left(1+\varepsilon^{1/n}\right)\right)$$

$$= 1 - \varepsilon^{1/n}$$

$$\lim_{n\to\infty} P\left(\left|\left|1-\frac{\theta}{\pi}\right|^n - 0\right| > \varepsilon\right) = 0 \Rightarrow \left|1-\frac{\theta}{\pi}\right|^n \xrightarrow{i.p.} 0$$

Hence, $\left|1-\frac{\theta}{\pi}\right|^n \xrightarrow{D} 0$

Exercise 6. $X_n \sim \text{Geom}\left(\frac{\lambda}{n}\right)$, $\lambda > 0$. $Y_n = \frac{X_n}{n}$. Show that Y_n converges to $Y \sim \text{Exp}(\lambda)$ in distribution.

Sol^n: $X_n \sim \text{Geom}\left(\frac{\lambda}{n}\right)$. Distribution of X_n,

$$P(X_n \leq m) = \sum_{i=1}^{m} \left(1-\frac{\lambda}{n}\right)^{i-1} \frac{\lambda}{n}$$

$$= \frac{\lambda}{n} \cdot \frac{1-\left(1-\frac{\lambda}{n}\right)^m}{\frac{\lambda}{n}} = 1-\left(1-\frac{\lambda}{n}\right)^m$$

Let us find distribution of Y_n,

$$P(Y_n \leq y) = P(X_n \leq ny) = 1-\left(1-\frac{\lambda}{n}\right)^{\lfloor ny \rfloor}$$

$$\lim_{n\to\infty} P(Y_n \leq y) = 1 - \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{\lfloor ny \rfloor} = 1 - e^{-\lambda y}$$

$$\Rightarrow Y_n \xrightarrow{D} Y \sim \text{Exp}(\lambda). \qquad \square$$