Tutorial 9

Monday, 26 September 2022 11:39 AM

Recap

1. P.W. $\Rightarrow$ a.s $\Rightarrow$ i.p $\Rightarrow$ D

   $L^p \Rightarrow$ i.p

   $L^p \Rightarrow L^q \ (1 \leq q < p)$

2. a.s. $\Longleftrightarrow \sum_{n \in \mathbb{N}} P(A_n(\varepsilon)) \quad \forall \varepsilon > 0$

3. <u>$\mathcal{L}^2$ Weak Law of Large Numbers</u>: (X_n) be a sequence of uncorrelated RVs with $\mathbb{E}X_n = \mu$, $Var(X_n) = \sigma^2$. Then $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ converges to μ in $\mathcal{L}^2$ and in probability.

4. <u>$\mathcal{L}^1$ Weak Law of Large Numbers</u>: (X_n) be a sequence of i.i.d RVs with $\mathbb{E}X_n = \mu$. Then $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ converges to μ in probability.

5. <u>$\mathcal{L}^4$ Strong law of Large Numbers</u>: (X_n) be a sequence of independent RVs with bounded mean and uniformly bounded 4^{th} central moment, i.e. $\sup_{n \in \mathbb{N}} \mathbb{E}(X_n - \mathbb{E}X_n)^4 \leq B < \infty$. Then $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ converges to $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}X_i$ almost surely.

6. <u>$\mathcal{L}^2$ Strong Law of Large Numbers</u>: (X_n) be a sequence of pairwise uncorrelated RVs with bounded mean and uniformly bounded variance, i.e., $\sup_{n \in \mathbb{N}} Var(X_n) \leq B < \infty$. Then $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ converges to $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}X_i$ almost surely.

7. <u>$\mathcal{L}^1$ Strong Law of Large Numbers</u>: (X_n) be sequence of RVs with uniformly bounded absolute mean, i.e., $\sup_{n \in \mathbb{N}} \mathbb{E}|X_n| \leq B < \infty$. Then $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ converges to

$\sup_{n\in\mathbb{N}} E|X_n| \le B < \infty$. Then $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ converges to $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} EX_i$ almost surely.

8. Central Limit Theorem : (X_n) be iid sequence of RVs with $EX_n = \mu$, $Var(X_n) = \sigma^2$ $\forall\, n \in \mathbb{N}$. Let $S_n = \sum_{i=1}^{n} X_i$. Then $\frac{S_n - n\mu}{\sigma\sqrt{n}}$ converges to $Y \sim N(0,1)$ in distribution.

Exercise 1 : a. Consider a sequence of iid Bernoulli RVs, (X_n) with parameter p. Define $S_n = \sum_{i=1}^{n} X_i$. State the results you can obtain from SLLN and CLT.

b. Find limiting distribution of $S_n = \sum_{i=1}^{n} X_i^n$ where $X_i^n \sim Ber(\lambda/n)$ for $1 \le i \le n$.

Sol^n a. SLLN: As $\sup_{n\in\mathbb{N}} E|X_n| = p < \infty$, $\frac{S_n}{n} \xrightarrow{a.s.} p$.

CLT: X_n's are i.i.d with mean p and variance $p(1-p)$. Hence

$$\frac{\sum_{i=1}^{n}(X_i - p)}{\sqrt{p(1-p)n}} \xrightarrow{D} Y \sim N(0,1)$$

b. We have that,

$$P(S_n = k) = \frac{n!}{k!(n-k)!} \cdot \frac{\lambda^k}{n^k} \cdot \left(1-\frac{\lambda}{n}\right)^{n-k}$$

Now, $$\lim_{n\to\infty} P(S_n = k) = \frac{\lambda^k}{k!} \lim_{n\to\infty} \frac{n}{n}\frac{(n-1)}{n}\cdots\frac{(n-k+1)}{n} \cdot \left(1-\frac{\lambda}{n}\right)^{n}\left(1-\frac{\lambda}{n}\right)^{-k}$$

$$= \frac{\lambda^k}{k!} \cdot 1 \cdot 1 \cdots 1 \cdot e^{\lambda} \cdot (1-0)^{-k}$$

$$= \frac{\lambda^k}{k!} \cdot e^{-\lambda}$$

$\therefore$ $S_n \xrightarrow{D} Y \sim Pois(\lambda)$

Exercise 2 A town starts a mosquito control program and the RV Z_n is the number of mosquitoes at the end of the n^{th} year $(n = 0, 1, \dots)$. Let X_n be the growth rate of mosquitoes in the year n, i.e., $Z_n = X_n Z_{n-1}$, $n \geq 1$. Assume that (X_n) is iid sequence of RVs with PMF

$$P(X=2) = \frac{1}{2}, \ P\left(X = \frac{1}{2}\right) = \frac{1}{4} \text{ and } P\left(X = \frac{1}{4}\right) = \frac{1}{4}.$$

Suppose that Z_0 is known and positive.

a. Find $\mathbb{E} Z_n$ and $\lim_{n \to \infty} \mathbb{E} Z_n$. What can you tell about success of the mosquito control program with the answer that you obtained?

b. $W_n = \log_2 X_n$. Find $\mathbb{E} W_n$ and $\mathbb{E} \log_2 \frac{Z_n}{Z_0}$.

c. Show that $\exists \ \alpha \in \mathbb{R}$ s.t. $\log_2 \frac{Z_n}{Z_0}$ converges to α almost surely.

d. Using c show that $Z_n \xrightarrow{a.s} \beta$ for some $\beta \in \mathbb{R}$ and evaluate β.

Sol^n:

a. $$Z_n = X_n X_{n-1} \cdots X_1 Z_0 = \left(\prod_{i=1}^{n} X_i\right) Z_0$$

Given that X_i's are iid, hence uncorrelated and Z_0 is fixed and known. Let $Z_0 = z_0$.

$$\therefore \ \mathbb{E} Z_n = (\mathbb{E} X_1)^n \cdot z_0$$

$$\mathbb{E} X_1 = 2 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4} = \frac{19}{16}$$

Therefore, $\mathbb{E} Z_n = \left(\frac{19}{16}\right)^n z_0$

So $\forall \ z_0 \neq 0$, $\lim \mathbb{E} Z_n = \infty$

So $\forall\ Z_0 \neq 0$, $\lim_{n\to\infty} \mathbb{E} Z_n = \infty$

otherwise, $\lim_{n\to\infty} \mathbb{E} Z_n = 0$

* We can't say nothing about $\lim_{n\to\infty} Z_n$ from this. Even $\mathbb{E}\left[\lim_{n\to\infty} Z_n\right]$ may not be equal to $\lim_{n\to\infty} \mathbb{E} Z_n$.

b. $$\mathbb{E} W_n = \mathbb{E} \log_2 X_n = \frac{1}{2} \cdot \log_2 2 + \frac{1}{4} \log \frac{1}{2} + \frac{1}{4} \log \frac{1}{4}$$
$$= \frac{1}{2} + \left(-\frac{1}{4}\right) + \left(-\frac{1}{2}\right)$$
$$= -\frac{1}{4}.$$

c. X_n's are i.i.d, hence, W_n's are iid. Consider the RV $\log_2 \frac{Z_n}{Z_0}$.

$$\log_2 \frac{Z_n}{Z_0} = \sum_{i=1}^{n} \log X_i = \sum_{i=1}^{n} W_i$$. It is easy to verify the condition for $\mathcal{L}^1$ SLLN and hence

$$\frac{1}{n} \log_2 \frac{Z_n}{Z_0} \xrightarrow{a.s.} \frac{1}{n} \cdot \frac{-n}{4} = -\frac{1}{4} \ (=: \alpha)$$

d. We know that if $a_n \to a$ then for any continuous function f, $f(a_n) \to f(a)$. Note that $f(x) = 2^x$ is continuous. N^c be complement of the exception set.

$$\therefore N^c = \left\{ \omega \in \Omega \,\middle|\, \lim_{n\to\infty} \frac{1}{n} \log\left(\frac{Z_n(\omega)}{Z_0(\omega)}\right) = -\frac{1}{4} \right\}$$
$$= \left\{ \omega \in \Omega \,\middle|\, \lim_{n\to\infty} \left(\frac{Z_n(\omega)}{Z_0(\omega)}\right)^{1/n} = 2^{-1/4} \right\}$$
$$= \left\{ \omega \in \Omega \,\middle|\, \lim_{n\to\infty} Z_n(\omega) = 0 \right\}$$

$\left[\lim_{n\to\infty} y = a,\ |a|<1 \Rightarrow \lim_{n\to\infty} y^n = 0 \right]$

As $\mathbb{P}(N^c) = 1$, $\lim_{n\to\infty} Z_n = 0$ almost surely.

As $P(N^c) = 1$, $\lim_{n\to\infty} Z_n = 0$ almost surely.

$\therefore \beta = 0$.

Exercise 3 : $[0,1]$ is partitioned into n disjoint subintervals with lengths $p_1, p_2, \dots, p_n$, and the entropy of this partition is defined to be

$$h = -\sum_{i=1}^{n} p_i \log p_i$$

Let $X_1, X_2, \dots$ be independent RVs having uniform dist. on $[0,1]$, and let $Z_m(i)$ be the number that $X_1, X_2, \dots X_m$ lie in the i^{th} interval of the partition above. Show that

$$R_m = \prod_{i=1}^{n} p_i^{Z_m(i)}$$

Satisfies $\frac{1}{m} \cdot \log R_m \xrightarrow{a.s.} -h$.

Sol^n:

We have that

$$Z_m(i) = \sum_{j=1}^{m} \mathbb{1}\{X_j \in I_i\} \quad \text{where } I_i \text{ is the } i^{th} \text{ interval.}$$

$$\log R_m = \sum_{i=1}^{n} Z_m(i) \log p_i = \sum_{i=1}^{n}\sum_{j=1}^{m} \mathbb{1}\{X_j \in I_i\} \log p_i$$

$$= \sum_{j=1}^{m}\sum_{i=1}^{n} \mathbb{1}\{X_j \in I_i\} \log p_i$$

We have that $\mathbb{E}\left[\mathbb{1}\{X_j \in I_i\}\right] = P(X_j \in I_i) = p_i$

Therefore, $\mathbb{E}\sum_{i=1}^{n} \mathbb{1}\{X_j \in I_i\} \log p_i = \sum_{i=1}^{n} p_i \log p_i = -h$

and,

$$\mathbb{E}\left[\log R_m\right] = -mh$$

By SLLN, $\frac{1}{m}\log R_m \to -h$ almost surely.

Exercise 4. Let $W_1, W_2, \ldots$ be a sequence of i.i.d., $\mathcal{N}(0,\sigma^2)$ RVs. Let $X_0 = 0$ and define

$$X_{n+1} = \frac{X_n + W_{n+1}}{2}, \quad n \geq 0.$$

Which RV does X_n converge in distribution to?

Sol^n:

$$X_n = \frac{X_{n-1} + W_n}{2} = \frac{W_n}{2} + \frac{1}{2}\left(\frac{X_{n-2} + W_{n-1}}{2}\right)$$

$$\vdots$$

$$= \frac{W_n}{2} + \frac{W_{n-1}}{2^2} + \cdots + \frac{W_1}{2^n}$$

Since $(W_i : i \in \mathbb{N})$ are i.i.d., $X_n \sim \mathcal{N}\left(0, \sigma^2\left(\sum_{i=1}^{n} 4^{-i}\right)\right)$

$$P(X_n \leq x) = P\left(\frac{X_n - 0}{\sqrt{\sigma^2 \sum_{i=1}^{n} 4^{-i}}} \leq \frac{x}{\sqrt{\sigma^2 \sum_{i=1}^{n} 4^{-i}}}\right)$$

$$= \phi\left(\frac{x}{\sqrt{\sigma^2 \sum_{i=1}^{n} 4^{-i}}}\right) \quad \forall\, x \in \mathbb{R}$$

Hence, $\lim_{n\to\infty} P(X_n \leq x) = \lim_{n\to\infty} F_{X_n}(x)$

$$= \phi\left(\lim_{n\to\infty} \frac{x}{\sqrt{\sigma^2 \sum_{i=1}^{n} 4^{-i}}}\right) \quad [\text{as } \phi \text{ is continuous}]$$

$$= \phi\left(\frac{\sqrt{3}\,x}{\sigma}\right)$$

$$\Rightarrow \quad X_n \xrightarrow{D} X \sim \mathcal{N}\left(0, \frac{\sigma^2}{3}\right)$$