Tractable Random Process :-

Recall:

Given, $(\Omega, \mathcal{F}, P)$, index set T and Statespace $\mathcal{X} \subseteq \mathbb{R}$

$$X : \Omega \longrightarrow \mathcal{X}^T$$

Characterised by,

$$F_{X_S} : \mathbb{R}^S \longrightarrow [0,1] \quad \forall \text{ finite } S \subseteq T.$$

$$\Rightarrow F_{X_S}(x_S) \triangleq P\left(\bigcap_{s \in S} X_s(\omega) \le x_s\right) \rightarrow \boxed{\text{for all } n \in \mathbb{N} \quad F_{X_1, \dots X_n}(x_1, \dots x_n)}$$

$$\left.\begin{aligned} m_X(t) &= \mathbb{E}[X_t] \\ R_X(t,s) &= \mathbb{E}[X_t X_s] \\ C_X(t,s) &= \mathbb{E}[X_t - m_X(t)][X_s - m_X(s)] \end{aligned}\right\} \Rightarrow \text{already done in previous tutorial.}$$

Example

(1) You have 100 Ruppees in an account with interest Rate 'R' compounded anually . (Find $m_X(n)$; $R_X(n,k)$; $C_X(n,k)$.

X_n is the value of amount in the account at year 'n'.

$$X_n = 100(1+R)^n$$

$R \sim$ Uniform $(0.04, 0.05)$

↳ Random Variable once it is fixed at the start it is fixed.

$$m_X(n) = \mathbb{E}[X_n] = 100\, \mathbb{E}[(1+R)^n]$$

$$= 100\, \mathbb{E}[(R')^n] \qquad R' \sim \text{uniform } (1.04, 1.05)$$

$= 100\, \mathbb{E}[(R')^n]$ $\qquad R' \sim \text{uniform}(1.04, 1.05)$

$= 100 \int_{1.04}^{1.05} 100 . (r')^n . \, dr'$

$= \frac{10^4}{n+1} (r')^{n+1} \Big|_{1.04}^{1.05}$

$= \frac{10^4}{n+1} ((1.05)^{n+1} - (1.04)^{n+1})$.

$R_X(n, K) = \mathbb{E}[X_n X_K] = 10^4\, \mathbb{E}[(1+R)^n (1+R)^K]$

$= 10^4\, \mathbb{E}[(R')^{n+k}]$

$= \frac{10^6}{n+k+1} [(1.05)^{m+n-1} - (1.04)^{m+n-1}]$

$C_X(n, K) = \mathbb{E}[[X_n - \mathbb{E}[X_n]][X_K - \mathbb{E}[X_K]]]$ $\longrightarrow$ calculate

⋮

(2) $X_t = A + Bt$

A and B $\sim \mathcal{N}(1,1)$, A and B are independent

$m_X(t) = \mathbb{E}[X_t]$

$= \mathbb{E}[A + Bt]$

$= \mathbb{E}[A] + E[Bt]$

$= 1 + t \quad \forall t \in [0, \infty)$

$R_X(t_1, t_2) = \mathbb{E}[X_{t_1} . X_{t_2}]$

$= \mathbb{E}[(A + Bt_1)(A + Bt_2)]$

$= \mathbb{E}[A^2 + ABt_2 + ABt_1 + B^2 t_1 t_2)]$

$= 2 + t_2 + t_1 + 2 t_1 t_2 \quad \forall t_1, t_2 \in [0, \infty)$

$C_X(t_1, t_2) =$

$1 + t_1 t_2$ (calculate)

Independent and identically distributed Process :-

$X : \Omega \longrightarrow \mathcal{X}^T$ is iid for any finite $S \subseteq T$ and $x_S \in \mathbb{R}^S$

for any finite $S \subseteq T$ and $x_s \in \mathbb{R}$

$$F_{X_S}(x_S) = P\left(\bigcap_{s\in S} \{X_s(\omega) \le x_s\}\right) \underset{\text{independence}}{=} \prod_{s\in S} F_{X_s}(x_s) \underset{\text{identical}}{=} \prod_{s\in S} F(x_s)$$

we can also write,

for any $n \in \mathbb{N}$

$$\longrightarrow F_{X_1, X_2, \dots X_n}(x_1, \dots x_n) \underset{\text{independent}}{=} \prod_{i=1}^{n} F_{X_i}(x_i) \underset{\text{identical}}{=} \prod_{i=1}^{n} F(x_i)$$

Remark: independent Random process and two Random process are independent are different concepts.

Stationary Process :-

$X: \Omega \longrightarrow X^T$ is stationary if for any t for all $n \in \mathbb{N}$

$$F_{X_S}(x_S) = F_{X_1, X_2, \dots X_n}(x_1, \dots x_n) \underset{\text{Stationary (shift invariant)}}{=} F_{X_1+t, X_2+t, \dots X_n+t}(x_1, \dots x_n)$$

iid $\implies$ stationary (was done in class)

stationary $\not\Rightarrow$ iid (Give an example)

$$\begin{cases} X_0 = X_0 \\ X_1 = X_0 \\ X_2 = X_1 \\ \vdots \\ X_n = X_{n-1} \end{cases} \quad \forall n \in \mathbb{N}$$

Remark :- Stationary Stochastic process $\implies$ All moments are shift invariant.

Weak or Wide Sense Stationarity :-

$\rightarrow \quad m_X(t) = m_X(t+s) \quad \forall\, s,t \in T$ and

$\rightarrow \quad R_X(t,s) = R_X(t+u, s+u) \quad \forall\, s,t,u \in T$

SSS $\implies$ WSS

but WSS $\not\Rightarrow$ SSS $\rightarrow$ Example

$$X : \Omega \longrightarrow \mathcal{X}^{\mathbb{N}}$$

$$X_{2n} = \begin{cases} -1 & \text{wp } 1/3 \\ 0 & \text{wp } 1/3 \\ 1 & \text{wp } 1/3 \end{cases} \quad \text{and} \quad X_{2n+1} = |X_{2n}| - 2/3$$

$R_X(n,m) = \mathbb{E}[X_n . X_m]$ if n and m are even

$= \mathbb{E}[X_n] . \mathbb{E}[X_m] = 0$ for n,m any other pairing.

$\mathbb{E}[X_{2n} . X_{2n+1}]$.

$= \mathbb{E}[X_{2n} . (|X_{2n}| - 2/3)]$

$= \mathbb{E}[X_{2n} . |X_{2n}|] - 2/3\, \mathbb{E}[X_{2n}]$

$= -1/3 + 0 + 1/3 - 2/3 . 0$

$= 0$

$\therefore$ this is WSS

$\therefore \quad R_X(n,m) = 0 \quad \forall\, n,m \in \mathbb{N}$

$$\left.\begin{aligned} P[(X_0, X_2) = (1,1)] &= 1/9 \\ P[(X_1, X_3) = (1,1)] &= 0 \end{aligned}\right\} \Rightarrow \text{"not Stationary"}$$

Random Walk

discrete time random process

Change 'X's' to S's,

$X = (X_n : n \in \mathbb{N})$

and U's $\rightarrow$ X's

$X_0 = 0$

$X_{n+1} = X_n + U_n$

$U = (U_n : n \in \mathbb{N})$ is iid

$U = (U_n : n \in \mathbb{N})$ is $\underline{\underline{\text{iid}}}$

$$X_t = X_t - X_0$$
$$= (X_t - X_s) + (X_s - X_0)$$
$$\Rightarrow \quad X_t - X_s = (X_t - X_0) - (X_s - X_0)$$
$$= \sum_{r=s}^{t-1} U_r$$

$\therefore$ $X_t - X_s$ depends on $U_s, \dots U_{t-1}$, as U is iid
$X_s - X_0$ depends on $U_0, \dots U_{s-1}$. these both are $\underline{\text{independent}}$.

(independent increament property of Random walk)

let $\mathbb{E}[U_1] = \mu$ and $\mathrm{Var}(U_1) = \sigma^2$

$$\mu_X(n) = \mathbb{E}[X_n] = \mathbb{E}[X_{n-1} + U_{n-1}] = \mathbb{E}[X_{n-1}] + \mu$$
$$= \mathbb{E}[X_{n-2}] + 2\mu$$
$$\vdots$$
$$= n\mu$$

$$R_X(n,m) = \mathbb{E}[X_n \cdot X_m]$$
$$= \mathbb{E}[(X_n - X_0)(X_m - X_0)]$$
$$= \mathbb{E}[(X_n - X_0)(X_m - X_n + X_n - X_0)]$$
$$= \mathbb{E}[(X_n - X_0)(X_m - X_n)] + \mathbb{E}[(X_n - X_0)^2]$$
$$= \mathbb{E}[(X_n - X_0)]\mathbb{E}[(X_m - X_n)] + \mathbb{E}[X_n^2]$$
$$= (n\mu)(m-n)\mu + n\sigma^2 + n^2\mu^2$$
$$= nm\mu^2 - \cancel{n^2\mu^2} + \cancel{n^2\mu^2} + n\sigma^2$$
$$= \underline{\underline{nm\mu^2 + n\sigma^2}}$$

$$R_X(n,n) = \mathbb{E}[X_n^2]$$

$$R_X(n,n) = \mathbb{E}[X_n^2]$$
$$= Var(X_n) + (\mathbb{E}[X_n])^2$$
$$= Var(X_{n-1} + U_{n-1}) + n^2 \mu^2$$
$$= Var(X_{n-1}) + \sigma^2 + n^2\mu^2$$
$$= n\sigma^2 + n^2\mu^2$$

$\therefore$ Random walk $\Rightarrow$ not WSS $\Rightarrow$ not stationary.

Levy Process :-

(L1) increaments are independent ; for any instants $0 \le t_1 < t_2 \dots < t_n < \infty$.

$X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \dots$ independent.

(L2) increaments are stationary.

(L3) continous in probability.

Markov process :-

$$\mathcal{F}_t = \sigma(X_s, s \le t)$$

for $u > t$

$$P(\{X_u \le x_u\} | \mathcal{F}_t) = P(\{X_u \le x_u\} | \sigma(X_t))$$

Example Random walk :-

$S : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$ is RW with

iid step size $\longrightarrow$ $X : \Omega \to \mathcal{X}^{\mathbb{N}}$ is Markov

$$P(\{S_{n+1} = y\} | \{S_n = x, S_{n-1} = s_{n-1}, \dots S_1 = s_1\})$$

increments are independent

$$= P(\{S_{n+1} - S_n = y - x\})$$
$$= P(\{S_{n+1} = y\} | \{S_n = x\})$$

Stopping times :-

$\mathcal{G}_\bullet \stackrel{?}{=} (\mathcal{G}_t \subseteq \mathcal{F} , t \in T)$ is called filtration is $\mathcal{G}_s \subseteq \mathcal{G}_t$ $\forall \; s \le t$.

"collection of event spaces"

def :- $\tau : \Omega \longrightarrow T$ is a stopping time with respect to $\mathcal{F}_\bullet$ if

(a) the event $\tau^{-1}[-\infty, t] \in \mathcal{F}_t$ for all $t \in T$

(b) $P\{\tau < \infty\} = 1$

Interpretation :- Given the history upto time 'n' of the process $\{X_m : m \in \mathbb{N}\}$, we can tell whether $\{T=n\}$ (or $\{T \le n\}$) has happened

$$P(T=n \mid \sigma(X_1, \dots X_n)) = 0 \text{ or } 1.$$

Example

(1) $N = \inf\{n \ge 1 : X_n \in A\} =$ (first hitting time of set A)

$$\{N > n\} = \{X_1 \notin A, \dots \; X_n \notin A\}$$

$$\Rightarrow \quad \{N > n\} \in \sigma(X_1, \dots X_n) \quad \forall \; n \ge 1$$

$$\Rightarrow \quad \{N \le n\} \in \sigma(X_1, \dots X_n) \quad \forall \; n \ge 1$$

$\therefore$ N is a stopping time.

Problem :- $X : \Omega \longrightarrow \{0,1\}^{\mathbb{N}}$ $\quad S : \Omega \longrightarrow \mathbb{Z}_+^{\mathbb{N}}$; $S_0 = 0$

let $A = \{1\}$

$\therefore$ $\tau_S^A := \inf\{n \in \mathbb{N} : S_n = 1\}$ is stopping time.

$$P(\tau_X^A = n) = P\left(\bigcap_{k=1}^{n-1} \{X_k = 0\} \cap \{X_n = 1\}\right)$$

$$= (1-p)^{n-1} p$$

$$P(\tau_X^A < \infty) = P\left(\bigcup_{n \in \mathbb{N}} \{\tau_X^A = n\}\right)$$

$n-1$

$$P(\tau_x^A < \infty) = P\left(\bigcup_{n \in \mathbb{N}} \{\tau_x^A = n\}\right)$$

$$= \sum P(\{\tau_x^A = n\}) = \sum (1-p)^{n-1} p = 1$$

Geometric distribution (PMF)

(2) let,

$$S_n = \sum_{k=1}^{n} X_k \quad ; \; n \geq 1 \qquad (X_n)_{n \in \mathbb{N}} \geq 0$$

$$N_t = \sup \{n \geq 1 : S_n \leq t\}$$

$$\{N_t = n\} = \{S_1 \leq t, S_2 \leq t, \ldots S_n \leq t, S_{n+1} > t, \ldots\}$$

$$= \bigcap_{k=1}^{n} \{S_k \leq t\} \cap \underbrace{\bigcap_{k=n+1}^{\infty} \{S_k > t\}}$$

depends on $X_{n+1}, X_{n+2}, \ldots$

$\therefore$ N_t is not a stopping time.

(3) $(X_n)_{n \in \mathbb{N}}$ be iid seq of Bernoulli RV

$$N_n := \sum_{k=1}^{n} \mathbb{1}_{\{X_k = 1\}} = \# \text{ succes upto time } n.$$

as $\sum_{k=1}^{n} \mathbb{1}_{\{X_k=1\}} \xrightarrow{n \to \infty} \infty$

$$T_m := \inf \{n \geq 1 : N_n = m\}$$

Then, $\forall\, n \geq 1$

$$\{T_m \leq n\} = \{N_n \geq m\} \longrightarrow \underline{\#\text{ of Succees at time 'n'} > \text{'m'}}$$

first time m-succes happen is less than 'n'

$$= \left\{\sum_{k=1}^{n} \mathbb{1}_{\{X_k = 1\}} \geq m\right\}$$

$$\in \sigma(X_1, \ldots X_k)$$

$\therefore$ T_m is a stopping time.

Wald's lemma :- Random walk : $S : \Omega \longrightarrow \mathbb{R}^{\mathbb{Z}_+}$ with iid step size $X : \Omega \to \mathbb{R}^{\mathbb{N}}$

$\mathbb{E}|X_1| < \infty$ and τ = finite mean stopping time.

Problem :- Consider $(X_n)_{n \geq 1}$.

$$P(X_n = 1) = \alpha$$

$$P(X_n = -1) = \beta$$

$i \in \mathbb{Z}$ such that $S_0 = i$

$$S_n := S_0 + \sum_{k=1}^{n} X_k$$

Let $\alpha < \beta$ and for some $j < i$

$$N := \inf\{n \in \mathbb{N} : S_n = j\}$$

Show that N is a stopping time ; and compute $\mathbb{E}[N]$

Solution :- $\mathbb{E}[X_1] = \alpha - \beta$

By SLLN

$$\frac{S_n - S_0}{n} = \frac{1}{n} \sum_{k=1}^{n} X_k \xrightarrow{as} \alpha - \beta < 0$$

$\Longrightarrow$ $\exists$ a set A such that $P(A) = 1$ for $w \in A$

$$\frac{S_n(w) - S_0}{n} \longrightarrow \alpha - \beta < 0 \quad \text{as } n \to \infty$$

$$\lim_{n \to \infty} S_n(w) - S_0 = -\infty \quad \forall\, w \in A.$$

$\therefore$ with probability '1' $S_n - S_0$ will cross any negative integer value in finite time.

$$\therefore \quad N(\omega) \leq n^*(\omega) < \infty$$

$$P(N(\omega) < \infty) = 1$$

and $\quad N := \inf\{ n \in \mathbb{N} : S_n = m \}$

$$\{N = n\} \in \sigma(X_1, \dots X_n).$$

$\therefore$ N is Stopping time.

$$\mathbb{E}[S_N - S_0] = \mathbb{E}\left[\sum_{k=1}^{N} X_k\right] = \mathbb{E}[N]\,\mathbb{E}[X_1]$$

$$\mathbb{E}[N] = \frac{\mathbb{E}[S_N] - \mathbb{E}[S_0]}{\mathbb{E}[X_1]} = \frac{j - i}{\alpha - \beta}$$ ✓