Tutorial 10

Monday, 26 September 2022 7:12 PM

DTMC: For a countable set $\mathcal{X}$, $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$ is called a DTMC if $\forall n \in \mathbb{Z}_+$, $\forall x, y \in \mathcal{X}$, and any historical event $H_{n-1} \cup \{X_n = x\}$

$$P(X_{n+1} = y \mid H_{n-1} \cup \{X_n = x\}) = P(X_{n+1} = y \mid X_n = x)$$

Example: Consider repeated tosses of a coin with a given probability of getting a head, $p \in (0,1)$. Let for $n \geq 0$, $X_n = 0$ if the outcome of the n^{th} toss is a tail. $X_n = 1$ otherwise.
(X_n) is i.i.d and Markov property holds trivially.

Exercise 1. Consider the event of infinite coin toss. The first coin is unbiased. Probability of head in the n^{th} toss is p if $n-1^{th}$ toss is head, else it is $1-p$.
Find $P(X_n = 1)$. Is (X_n) Markov? Find the transition probability matrix. Draw the transition graph.
Find the distribution of X_3 given $X_1 = 1$.

Sol^n:

a. $$P(X_1 = 1) = P(X_1 = 1 \mid X_0 = 1) P(X_0 = 1) + P(X_1 = 1 \mid X_0 = 0) P(X_0 = 0)$$
$$= p \cdot \tfrac{1}{2} + (1-p) \cdot \tfrac{1}{2} = \tfrac{1}{2}$$

Let $P(X_{n-1} = 1) = \frac{1}{2}$, then

$$P(X_n = 1) = P(X_n = 1 \mid X_{n-1} = 1) P(X_{n-1} = 1) + P(X_n = 1 \mid X_{n-1} = 0) P(X_{n-1} = 0)$$
$$= p \cdot \tfrac{1}{2} + (1-p) \cdot \tfrac{1}{2} = \tfrac{1}{2}$$

So by principle of mathematical induction.

So by principle of mathematical induction,

$$P(X_n = 1) = \frac{1}{2} \quad \forall \; n \in \mathbb{N}.$$

b. $P(X_n = 1 \mid H_{n-1}) = P(X_n = 1 \mid X_1 = x_1, X_2 = x_2, \ldots, X_{n-1} = x_{n-1})$

$$= P(X_n = 1 \mid X_{n-1} = x_{n-1})$$

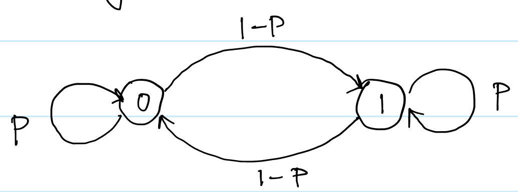
$$= \begin{cases} p & \text{if } x_{n-1} = 1 \\ 1-p & \text{otherwise} \end{cases}$$

c. Transition probability matrix,

$$P = \begin{bmatrix} P(X_n = 0 \mid X_{n-1} = 0) & P(X_n = 1 \mid X_{n-1} = 0) \\ P(X_n = 0 \mid X_{n-1} = 1) & P(X_n = 1 \mid X_{n-1} = 1) \end{bmatrix}$$

$$= \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

d. Transition graph,



e. Given that distribution of X_1, $\pi_1 = [0 \quad 1]$. By Chapman-Kolmogorov,

$$\pi_3 = \pi_1 P^2 = [0 \quad 1] \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix} \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

$$= \left[2(1-p)p \quad (1-p)^2 + p^2 \right]$$

Exercise 2: Consider iid coin tosses with probability of head, p. Define $Y_0 = 0$ and for $n \geq 1$, $Y_n = \sum_{i=0}^{n-1} X_i$, i.e. Y_n is the number of heads until n^{th} toss. Is Y_n Markov? Find transition probability matrix. Draw transition graph.

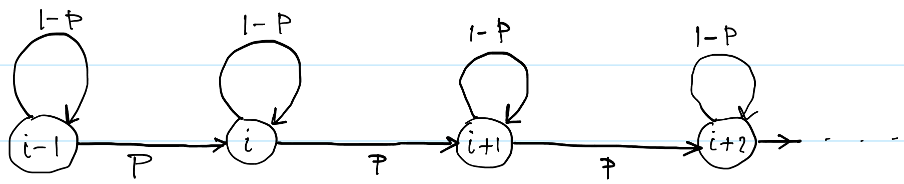
Soln :

a. $P(Y_{n+1} = j \mid Y_0 = i_0, \ldots, Y_{n-1} = i) = \begin{cases} p & \text{if } j = i+1 \\ 1-p & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$

$= P(Y_{n+1} = j \mid Y_n = i)$

b. $$P = \begin{bmatrix} 1-p & p & 0 & 0 & \cdots \\ 0 & 1-p & p & 0 & \cdots \\ 0 & 0 & 1-p & p & \cdots \\ \vdots & 0 & 0 & 1-p & \cdots \\ & \vdots & & \vdots & \end{bmatrix}$$

c.



Note: n is a deterministic time. What if we condition X_T on X_{T-1} where T is random ?

Exercise 3 : There are N empty boxes and an infinite collection of balls. At each step a box is chosen at random and a ball is placed in it. Let X_n be the number of empty boxes after the n^{th} ball is placed.

a. Show that (X_n) is a DTMC.

b. Find transition probabilities.

c. Write the transition probability matrix for $N = 3$

Soln :

a,b $X_{n+1} = X_n + Z_n$ where

$$Z_n = \begin{cases} 0 & \text{w.p. } \frac{N - X_n}{N} \\ -1 & \text{w.p. } \frac{X_n}{N} \end{cases}$$

$$\therefore \quad P(X_{n+1} = i \mid X_n = j, H_{n-1}) = \begin{cases} \frac{N-j}{N} & \text{if } i = j \\ \frac{j}{N} & \text{if } i = j-1 \\ 0 & \text{otherwise.} \end{cases}$$

$$= P(X_{n+1} = i \mid X_n = j)$$

c

$$P(X_{n+1} = i \mid X_n = 3) = \begin{cases} 1 & \text{if } i = 2 \\ 0 & \text{o.w.} \end{cases}$$

$$P(X_{n+1} = i \mid X_n = 2) = \begin{cases} \frac{1}{3} & \text{if } i = 2 \\ \frac{2}{3} & \text{if } i = 1 \\ 0 & \text{o.w.} \end{cases}$$

$$P(X_{n+1} = i \mid X_n = 1) = \begin{cases} \frac{2}{3} & \text{if } i = 1 \\ \frac{1}{3} & \text{if } i = 0 \\ 0 & \text{o.w.} \end{cases}$$

$$P(X_{n+1} = i \mid X_n = 0) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{o.w.} \end{cases}$$

$$\therefore \quad P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Exercise 4 : Let (X_n) be a homogeneous Markov chain with $P(X_{n+1} = k \mid X_n = j) = p_{jk}$. Let z^i the first time instance when $X_{z_i} = i$. Find

a. $P(X_{z_i+1} = k \mid X_{z_i} = i)$

b. $P(X_{z_i} = k \mid X_{z_i-1} = j)$

b. $P\left(X_{\tau_i} = k \mid X_{\tau_i - 1} = j\right)$

<u>Sol^n</u>: a. $P\left(X_{\tau_{i+1}} = k \mid X_{\tau_i} = i\right) = p_{ik}$

b. $P\left(X_{\tau_i} = k \mid X_{\tau_i - 1} = j\right) = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{otherwise.} \end{cases}$

* Is τ_i a stopping time? Is $\tau_i - 1$ a stopping time?

<u>Exercise 5</u>: Let (S_n) be a simple random walk with $S_0 = 0$, show that $X_n = |S_n|$ is Markov Chain; find the transition probabilities of the Chain.

Let $M_n = \max\{S_k : 0 \le k \le n\}$, and show that $X_n = M_n - S_n$ defines a Markov chain.

<u>Sol^n</u>: $X_{n+1} \in \{i-1, i+1\}$ if $X_n = i$, $n \in \mathbb{Z}_+$, H_{n-1} is history.

<u>a</u> $P\left(X_{n+1} = i+1 \mid X_n = i \cap H_{n-1}\right) =$

$$P\left(X_{n+1} = i+1 \mid S_n = i \cap H_{n-1}\right) P\left(S_n = i \mid X_n = i\right)$$

$$+ P\left(X_{n+1} = i+1 \mid S_n = -i \cap H_{n-1}\right) \cdot P\left(S_n = -i \mid X_n = i\right)$$

$$= P\left(S_n - S_{n-1} = 1\right) \cdot \frac{P\left(S_n = i\right)}{P\left(X_n = i\right)} + P\left(S_n - S_{n-1} = -1\right) \cdot \frac{P\left(S_n = -i\right)}{P\left(X_n = i\right)}$$

Let $\ell = \max\{t : S_t = 0, t \le n\}$; $\ell \geq 0$ as $S_0 = 0$

$$P(S_n = i) = P\left(\{S_n = i\} \cap \{S_\ell = 0\}\right) \quad \text{as} \quad P(S_\ell = 0) = 1$$

$$= P\left(S_n = i \mid S_\ell = 0\right) P\left(S_\ell = 0\right)$$

$$= P\left(S_n = i \mid S_\ell = 0\right)$$

$$= P(S_n = i \mid S_\ell = 0)$$

$$\therefore \; P(S_n = i) = p^{n_1} \cdot (1-p)^{n_2}$$

where $n_1 - n_2 = i$ and $n_1 + n_2 = n - \ell$

$$\therefore \; n_1 = \frac{1}{2}(n-\ell) + \frac{i}{2} \quad , \; n_2 = \frac{1}{2}(n-\ell) - \frac{i}{2}$$

Similarly we get $P(S_n = -i) = p^{n_2}(1-p)^{n_1}$

$$\therefore \; \frac{P(S_n = i)}{P(X_n = i)} = \frac{P(S_n = i)}{P(S_n = i) + P(S_n = -i)}$$

$$= \frac{p^{\frac{1}{2}(n-\ell)+\frac{i}{2}} (1-p)^{\frac{1}{2}(n-\ell)-\frac{i}{2}}}{p^{\frac{1}{2}(n-\ell)+\frac{i}{2}} (1-p)^{\frac{1}{2}(n-\ell)-\frac{i}{2}} + p^{\frac{1}{2}(n-\ell)-\frac{i}{2}} (1-p)^{\frac{1}{2}(n-\ell)+\frac{i}{2}}}$$

$$= \frac{p^{i/2}(1-p)^{-i/2}}{p^{i/2}(1-p)^{-i/2} + p^{-i/2}(1-p)^{i/2}}$$

$$= \frac{p^i}{p^i + (1-p)^i}$$

Similarly, $\dfrac{P(S_n = -i)}{P(X_n = i)} = \dfrac{(1-p)^i}{p^i + (1-p)^i}$

$$\therefore \; P(X_n = i+1 \mid X_n = i \cap H_{n-1}) = p \cdot \frac{p^i}{p^i + (1-p)^i} + (1-p)\frac{(1-p)^i}{p^i + (1-p)^i}$$

$$= \frac{p^{i+1} + (1-p)^{i+1}}{p^i + (1-p)^i}$$

$$\therefore \; P(X_n = i-1 \mid X_n = i \cap H_{n-1}) = 1 - \frac{p^{i+1} + (1-p)^{i+1}}{p^i + (1-p)^i}$$

Finally, $P(X_{n+1} = 1 \mid X_n = 0) = 1$

b. If $Y_n > 0$, $M_{n+1} = M_n$ $\therefore Y_{n+1} \in \{Y_n - 1, Y_n + 1\}$

If $Y_n = j \neq 0$

$$Y_{n+1} = \begin{cases} j-1 & \text{w.p. } p \\ j+1 & \text{w.p } 1-p \end{cases}$$

If $Y_n = 0 \Rightarrow M_n = S_n$

$$Y_{n+1} = \begin{cases} 0 & \text{w.p. } p \\ 1 & \text{w.p } 1-p \end{cases}$$

* Is M_n Markov?
* Is sum of two Markov processes Markov?