Tutorial 13

Monday, 24 October 2022 8:56 AM

Communicating classes :

- $i \to j$ if $p_{ij}^{(n)} > 0$ for some $n \in \mathbb{Z}_+$ (j is accessible from i)

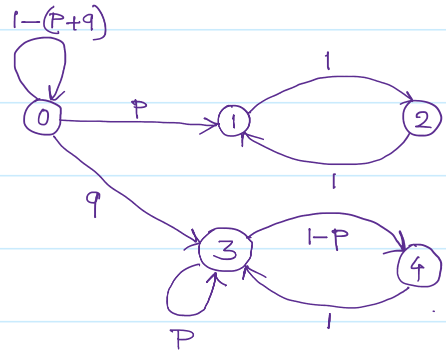
- $i \leftrightarrow j$ if $i \to j$ and $j \to i$ (i, j communicate with eachother)

- Communication is an equivalence relation i.e.

  1. $x \leftrightarrow x$
  2. $x \leftrightarrow y \Rightarrow y \leftrightarrow x$
  3. $x \leftrightarrow y$ and $y \leftrightarrow z \Rightarrow x \leftrightarrow z$.

- An equivalence relation partitions the space. Here the state space is partitioned into communicating classes.

Example 1: Consider the DTMC with following transition graph:



Assume $p, q \in (0,1)$. What are the communicating classes in the DTMC? Can you find some common properties of each of the communicating classes?

* It is easy to see that -

  $0 \to 1$, $0 \to 2$, $0 \to 3$, $0 \to 4$.

  $1 \to 2$, $2 \to 1$

$3 \to 4$, $4 \to 3$

So, the communicating classes are $\{0\}, \{1,2\}, \{3,4\}$

· Transient State: State 'i' is called transient if $\sum_{n \in \mathbb{N}} f_{ii}^{(n)} = f_{ii} < 1$.

· Recurrent State: State 'i' is called recurrent if $f_{ii} = 1$.

· 'i' is positive recurrent if $\sum_{n \in \mathbb{N}} n f_{ii}^{(n)} < \infty$

· 'i' is null recurrent if $\sum_{n \in \mathbb{N}} n f_{ii}^{(n)} = \infty$

· Closed communicating class: A communicating class C is closed if $\forall\, i,j \in C \times C^c$, $p_{ij} = 0$.

Otherwise C is open.

· Period of a state: Period of state i, $d_i = \gcd T_i$ where $T_i = \{n \in \mathbb{N} : p_{ii}^{(n)} > 0\}$.

'i' is aperiodic if $d_i = 1$.

* In the example 1, $f_{00} = 1 - (p+q)$, hence 0 is transient state.

$f_{11}^{(2)} = 1 \Rightarrow f_{11} = 1$, hence 1 is recurrent state.

Similarly 2 is recurrent state.

$$f_{ii}^{(n)} = \begin{cases} 1 & \text{if } n = 2 \\ 0 & \text{Otherwise} \end{cases} \quad \text{for } i \in \{1,2\}$$

Mean recurrence time, $\mu_i = 2$ for $i \in \{1,2\}$

$f_{33}^{(1)} = p$, $f_{33}^{(2)} = 1-p \Rightarrow f_{33} = 1$, hence 3 is recurrent state.

Mean recurrence time, $\mu_3 = p + 2(1-p) = 2-p$

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$f_{44}^{(1)} = 0, \quad f_{44}^{(2)} = 1-p, \quad f_{44}^{(3)} = p(1-p), \quad f_{44}^{(4)} = p^2(1-p), \ldots$

$$f_{44}^{(n)} = \begin{cases} (1-p)\,p^{n-2} & \text{for } n \geq 2 \\ 0 & \text{otherwise.} \end{cases}$$

Mean recurrence time, $\mu_4 = \sum_{n \geq 2} (1-p)\, p^{n-2} \cdot n = \frac{2-p}{1-p}$

* $\{0\}$ is open, $\{1,2\}$ and $\{3,4\}$ are closed.

* $\mathcal{T}_0 = \{1, 2, \ldots\} \Rightarrow d_0 = 1.$

  $\mathcal{T}_1 = \{2, 4, \ldots\} \Rightarrow d_1 = 2$

  $\mathcal{T}_2 = \{2, 4, \ldots\} \Rightarrow d_2 = 2.$

  $\mathcal{T}_3 = \{1, 2, \ldots\} \Rightarrow d_3 = 1$

  $\mathcal{T}_4 = \{2, 3, \ldots\} \Rightarrow d_4 = 1$

Results:

1. Transience, Recurrence are class properties.
2. Periodicity is class property.

Theorem: Open communicating classes are transient.

Proof: If C is an open communicating class, $\exists$ $j \notin C$ and $i \in C$ such that $p_{ij} > 0$.

Also $f_{ji} = 0$, otherwise $j \to i$ and hence $j \in C$.

Now,
$$f_{ii} = p_{ii} + \sum_{k,\, k \neq i} p_{ik} f_{ki}$$
$$= p_{ii} + \sum_{\substack{k,\, k \neq i \\ k \neq j}} p_{ik} f_{ki} + p_{ij} f_{ji}$$
$$= p_{ii} + \sum_{k:\, k \neq i,\, k \neq j} p_{ik} f_{ki}$$

$$= p_{ii} + \sum_{k: k\neq i, k\neq j} p_{ik} f_{ki}$$

$$\leq p_{ii} + \sum_{\substack{k: k\neq i \\ k\neq j}} p_{ik} = 1 - p_{ij} < 1$$

Hence, i is transient and hence class C is transient.

<u>Corollary</u>: Recurrent communicating classes are closed.

<u>Results</u>.

1. $\lim\limits_{n\to\infty} p_{jj}^{(n)} = 0$ if j is transient.

2. $\sum\limits_{n\in\mathbb{N}} p_{jj}^{(n)} < \infty$ if j is transient.

3. $$\lim_{n\to\infty} \frac{1}{n} \sum_{n\in\mathbb{N}} p_{jj}^{(n)} = \begin{cases} 0 & \text{if } j \text{ is null recurrent.} \\ \frac{1}{\mu_{jj}} & \text{if } j \text{ is positive recurrent.} \end{cases}$$

<u>Theorem</u>: Finite closed communicating classes are positive recurrent.

<u>Proof</u>: Let C be a closed, finite communicating class.

If $i \in C$, then $P(\{X_n \in C\} | \{X_0 = i\}) = 1$, i.e.

$$\sum_{j\in C} p_{ij}^{(n)} = 1$$

If C were transient or, null recurrent,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} p_{ij}^{(n)} = 0$$

Therefore,

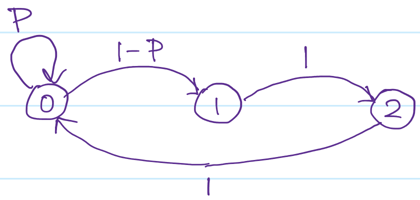
$$\sum_{j\in C} \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} p_{ij}^{(n)} = 0$$

Now, $$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{j\in C} p_{ij}^{(n)} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} 1 = 1 \quad \Rightarrow\!\Leftarrow$$

Hence C is positive recurrent.

<u>Irreducible Markov Chain</u>: A Markov chain with single class is called irreducible Markov chain.

<u>Example 2</u>: Consider the DTMC with following transition graph.



- How many communicating classes are there?
- Identify whether they are recurrent or transient.

- $0 \to 1$, $1 \to 2$, $2 \to 0$, hence $\exists$ only one communicating class.
- finite, closed communicating class, hence positive recurrent.

Let P be the transition probability matrix of the time-homogeneous Markove chain $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$.

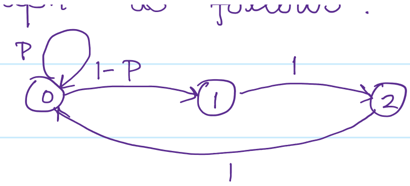
<u>Invariant Distribution</u>: A probability distribution π on the state space is called invariant distribution for the Markov chain if it satisfy the global balance equation.

$$\pi = \pi P$$

- P is a stochastic matrix, π is its left eigenvector corresponding to eigen value 1.

<u>Example 3</u> Find invariant distribution of the DTMC with transition graph as follows:

transition graph as follows.



$$P = \begin{bmatrix} p & 1-p & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Global balance equation :

$$[\pi_1 \ \ \pi_2 \ \ \pi_3] = [\pi_1 \ \ \pi_2 \ \ \pi_3] \begin{bmatrix} p & 1-p & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \pi_1 = \pi_1 p + \pi_3 \quad , \quad \pi_2 = (1-p)\pi_1 \quad , \quad \pi_3 = \pi_2$$

$$\therefore \quad \pi = \left[\frac{x}{1-p} \quad x \quad x \right] \quad \text{where} \quad 2x + \frac{x}{1-p} = 1$$

$$\Rightarrow 3x - 2px = 1$$

$$\Rightarrow x = \frac{1}{3-2p}$$

$$\Rightarrow \quad \pi = \left[\frac{1}{(1-p)(3-2p)} \quad \frac{1}{3-2p} \quad \frac{1}{3-2p} \right]$$

<u>Ergodic Markov Chain</u> : A Markov chain is called ergodic if it is irreducible, aperiodic, positive recurrent.

- For ergodic Markov chains,

$$\pi(j) = \lim_{n\to\infty} P_{ij}^{(n)} \quad - \text{Steady state probability.}$$

<u>Ex1</u> : Let there are two urns and each urn has 5 balls. Out of total 10 balls 5 are white and 5 are black. At each time 1 ball is drawn from each of the urns and exchanged. Let X_n takes the number of white balls in urn 1.

a. Show that $\{X_n\}$ is a Markov chain.

b. Find the steady state probability that the number of white balls in the 1st urn be zero.

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Sol^n State space, $\mathcal{X} = \{0, 1, \ldots, 5\}$. Let the current state be $x \in \mathcal{X}$. 4 cases are possible -

① White from urn 1, White from urn 2 : Prob - $\frac{x}{5} \cdot \frac{5-x}{5}$

② White from urn 1, Black from urn 2 : Prob - $\frac{x}{5} \cdot \frac{x}{5}$

③ Black from urn 1, White from urn 2 : Prob - $\frac{5-x}{5} \cdot \frac{5-x}{5}$

④ Black from urn 1, Black from urn 2 : Prob - $\frac{5-x}{5} \cdot \frac{x}{5}$

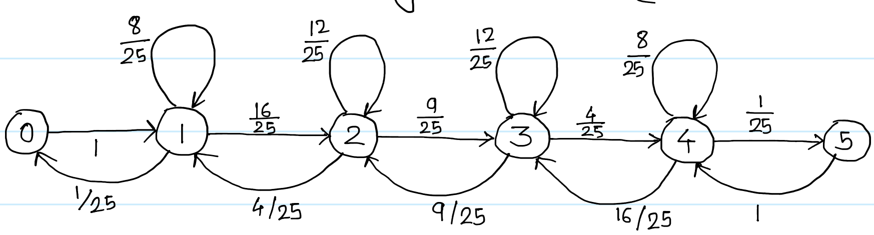
If 1 or 4 happens, $X_{n+1} = X_n$

If 2 happens, $X_{n+1} = X_n - 1$

If 3 happens, $X_{n+1} = X_n + 1$

$$X_{n+1} = \begin{cases} X_n & \text{w.p. } \dfrac{2(5-X_n)X_n}{25} \\ X_n + 1 & \text{w.p. } \dfrac{(5-X_n)^2}{25} \\ X_n - 1 & \text{w.p. } \dfrac{X_n^2}{25} \end{cases}$$

So, by random mapping theorem $\{X_n\}$ is a Markov chain.



So, the transition probability matrix

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{25} & \frac{8}{25} & \frac{16}{25} & 0 & 0 & 0 \\ 0 & \frac{4}{25} & \frac{12}{25} & \frac{9}{25} & 0 & 0 \\ 0 & 0 & \frac{9}{25} & \frac{12}{25} & \frac{4}{25} & 0 \\ 0 & 0 & 0 & \frac{16}{25} & \frac{8}{25} & \frac{1}{25} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

* Find π that satisfies the equation $\pi = \pi P$.

$\pi(0)$ is the steady state probability of 'urn 1 having no white ball'.