

TA Session 14

Poisson Process

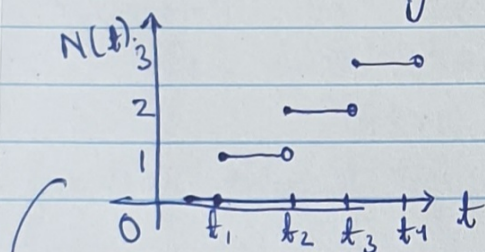
Recap of important definitions and results

- ① Defn: A simple point process is a random countable collection of distinct points $S: \Omega \rightarrow \mathcal{X}^N$ s.t. $\|S_n\| \rightarrow \infty$ as $n \rightarrow \infty$

Simple point process on the half-line:- (arrival process)
 $\tilde{S}: \Omega \rightarrow \mathbb{R}_+^N$ is obtained by ordering the points of the process $S: \Omega \rightarrow \mathcal{X}^N$ where $\tilde{S}_n = S_{(n)}$ is the n th order statistic of S
 $\tilde{S}_0 \triangleq 0, \quad \tilde{S}_n \triangleq \inf \{t \in [0, \infty) : N(t) \geq n\}$

- ② Defn: A stochastic process, $N: \Omega \rightarrow \mathbb{Z}_+^R$, associated with is said to be a counting process if:

- (i) $N(t) \geq 0 \quad \forall t \in \mathbb{R}_+$,
- (ii) $N(s) \leq N(t)$, for $s < t$; $s, t \in [0, \infty)$ (monotonically non-decreasing)
- (iii) $N(\cdot)$ is right continuous function of t .



A counting process is called simple if

$$P(\{\omega: \text{all jumps of } N(t, \omega) \text{ are of unit size}\}) = 1.$$

→ Sample path of simple counting process

- ③ Defn: A counting process associated with a simple point process is a random process $\{N(t), t \geq 0\}$.

$$N(t) = \sum_{n \in \mathbb{N}} 1_{\{\tilde{S}_n \leq t\}}$$

where $N(t)$ records the # of events / arrivals that have happened in the interval $(0, t]$

If $\{\tilde{S}_n: \Omega \rightarrow \mathbb{R}_+^N$ is a simple point process, then $\{N(t), t \geq 0\}$ is said to be a simple counting process.

④ Defn: A counting process $\{N(t), t \geq 0\}$ is said to be a ^{homogeneous} Poisson process of rate $\lambda, \lambda > 0$ if

(i) $N(0) = 0$

(ii) the process has independent increments

(iii) the process has stationary increments.

(iv) $N(t)$ has Poisson pmf i.e. $P(N(t) = n) = \frac{e^{-\lambda t} (\lambda t)^n}{n!}$

⑤ A counting process $\{N(t): t \geq 0\}$ is said to have independent increments if:

$\forall 0 = t_0 < t_1 < t_2 < \dots < t_k$, where $k \geq 1$ is any integer, the collection of random variables

$\{N(t_i) - N(t_{i-1}): 1 \leq i \leq k\}$ is mutually independent

⑥ A counting process $\{N(t): t \geq 0\}$ is said to have stationary increments if for all $s, t \in \mathbb{R}_+$

the distribution of the random variable

$N(t+s) - N(t)$ does not depend on t but is a function only of s .

i.e. the distribution of no. of arrivals that occur in an interval depends only on width of the interval.

⑦ Simple counting process $N: \Omega \rightarrow \mathbb{Z}_+^{\mathbb{R}_+}$ and arrival process $\tilde{S}: \Omega \rightarrow \mathbb{R}_+^{\mathbb{N}}$ are inverse processes:

$$\boxed{\{\tilde{S}_n \leq t\} = \{N_t \geq n\}}$$

Exercise 1:- Given that $N(t)$ $N: \Omega \rightarrow \mathbb{Z}_+^{R_+}$ is a Poisson process, what is the distribution of the associated point process $\tilde{S}: \Omega \rightarrow \mathbb{R}_+^N$?

Soln:- $F_{\tilde{S}_n}(t) = P(\tilde{S}_n \leq t) = P(N(t) \geq n)$ [from inverse relation]

$$= \sum_{k=n}^{\infty} \frac{e^{-\lambda t} (\lambda t)^k}{k!}$$

[by defn. of Poisson Process]

$$\Rightarrow \boxed{f_{\tilde{S}_n}(t) = \frac{dF_{\tilde{S}_n}(t)}{dt} = \frac{\lambda e^{-\lambda t} (\lambda t)^{n-1}}{(n-1)!}, t \in [0, \infty)}$$

$\rightarrow$ gamma distribution (n, λ)

Exercise 2:- Compute the finite dimensional distribution of the counting process.

Soln:- $P(N(t_1)=n_1, N(t_2)=n_2, \dots, N(t_k)=n_k), t_1 < t_2 < \dots < t_k$

$$= P(N(t_1)=n_1, N(t_2)-N(t_1)=n_2-n_1, \dots, N(t_k)-N(t_{k-1})=n_k-n_{k-1})$$

$$= P(N(t_k)-N(t_{k-1})=n_k-n_{k-1} \mid N(t_{k-1})-N(t_{k-2})=n_{k-1}-n_{k-2}, \dots, N(t_1)=n_1)$$

$$P(N(t_{k-1})-N(t_{k-2})=n_{k-1}-n_{k-2}, \dots, N(t_1)=n_1)$$

$$= P(N(t_1)=n_1) \cdot P(N(t_2)-N(t_1)=n_2-n_1 \mid N(t_1)=n_1) \cdot \dots \cdot P(N(t_k)-N(t_{k-1})=n_k-n_{k-1} \mid \dots)$$

* $P(N(t_2)-N(t_1)=n_2-n_1 \mid N(t_1)=n_1) = P(N(t_2)-N(t_1)=n_2-n_1)$

↑
Independent increment property

$$\begin{aligned}
 * \quad P(N(t_2) - N(t_1) = n_2 - n_1) &= P(N(t_2 - t_1) - N(t_1 - t_1) = n_2 - n_1) \\
 &\quad \uparrow \\
 &\quad \text{Stationary Increment property} \\
 &= P(N(t_2 - t_1) - N(0) = n_2 - n_1) \\
 &= P(N(t_2 - t_1) = n_2 - n_1) = \frac{e^{-\lambda(t_2 - t_1)} (\lambda(t_2 - t_1))^{(n_2 - n_1)}}{(n_2 - n_1)!}
 \end{aligned}$$

$$\begin{aligned}
 \text{Hence, } P(N(t_1) = n_1, N(t_2) = n_2, \dots, N(t_k) = n_k) \\
 = \left(\frac{e^{-\lambda t_1} (\lambda t_1)^{n_1}}{n_1!} \right) \left(\frac{e^{-\lambda(t_2 - t_1)} (\lambda(t_2 - t_1))^{(n_2 - n_1)}}{(n_2 - n_1)!} \right) \dots \left(\frac{e^{-\lambda(t_k - t_{k-1})} (\lambda(t_k - t_{k-1}))^{(n_k - n_{k-1})}}{(n_k - n_{k-1})!} \right)
 \end{aligned}$$

⑧ Alternate Defn. of Poisson Process - (In terms of inter-arrival times)

A simple counting process $N: \Omega \rightarrow \mathbb{Z}_+^{R_+}$ is a homogeneous Poisson process with a finite +ve rate λ , iff the inter-arrival time sequence $X: \Omega \rightarrow \mathbb{R}_+^N$ are iid random variables with an exponential distribution of rate λ .

$$\begin{aligned}
 \text{inter-arrival time } X_n = \tilde{S}_n - \tilde{S}_{n-1} \quad \text{with } X_0 = 0. \\
 \text{where } \tilde{S}_n \text{ is } n\text{th arrival instant.} \\
 \Leftrightarrow \tilde{S}_n = \sum_{i=1}^n X_i, \quad n \in \mathbb{N}.
 \end{aligned}$$

Why are X_i 's iid?

→ the independence follows from the strong independent increments property of the Poisson process.

→ the identical distribution follows from the strong stationary increments property (in case of homogeneous Poisson Process)

Exercise 3 — let $N: \Omega \rightarrow \mathbb{Z}_+^{R+}$ be a Poisson process of rate $\lambda = 2$.
let x_i denote the i th interarrival time.

(i) Find $P(x_1 > 0.4)$.

(ii) given there were no arrivals before $t=1$, compute $P(x_1 > 3 \mid \text{no arrivals before } t=1)$.

(iii) given the 3rd arrival occurred at $t=2$, find the prob. that 4th arrival occurs after $t=4$.

Soln: — (i) $P(x_1 > 0.4) = e^{-\lambda(0.4)} = e^{-0.8}$

(ii) $N(1) = 0$

$$P(x_1 > 3 \mid N(1) = 0) = P(N(3) = 0 \mid N(1) = 0)$$

$$= P(N(3) - N(1) = 0 \mid N(1) = 0)$$

$$= P(N(3) - N(1) = 0)$$

$$\begin{array}{c} \uparrow \text{IIP} \\ \uparrow \text{SIP} \end{array} \quad \begin{array}{c} = \\ \uparrow \end{array} P(N(2) = 0) = e^{-2 \cdot 2} = e^{-4}$$

$(x_1 > 3) \Rightarrow 1^{\text{st}} \text{ arrival happens after } t=3$

$$\Rightarrow (x_1 > 3) = (N_3 < 1) = (N_3 = 0)$$

(iii) $P(x_4 > 4 \mid N(2) = 3)$

$$\begin{array}{c} \uparrow \text{IIP} \end{array} \quad = P(N(4) - N(2) = 0 \mid N(2) = 3)$$

$$= P(N(4) - N(2) = 0)$$

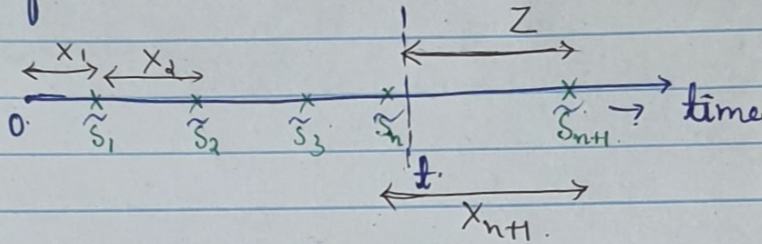
$$\begin{array}{c} \uparrow \text{SIP} \end{array} \quad = P(N(2) = 0)$$

$$= e^{-4}$$

Exercise 4:- Consider a Poisson Process of rate λ and fix any $t > 0$. The length of time interval from t until first arrival after t is a non-ve RV Z .

- (i) Find the distribution of Z given $N(t) = 0$
 (ii) Show that Z is independent of all arrival instants and number of arrivals till t , $\{N(z), z \leq t\}$.

Soln:-

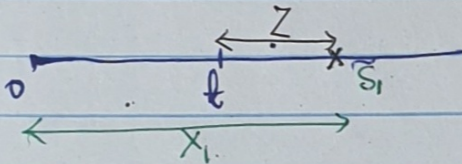


- (i) When $\{N(t) = 0\}$ i.e. no arrivals till time t ,

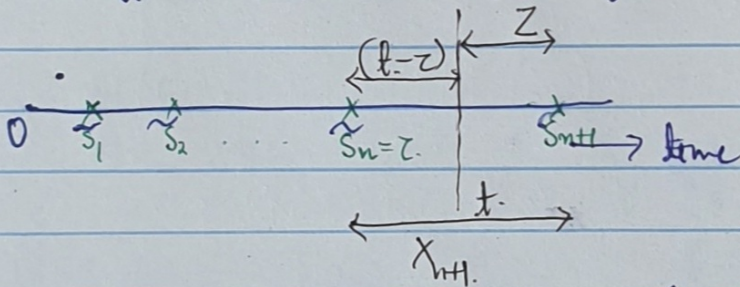
$$= \{X_1 > t\}$$

$$P(\{Z > z\} | \{N(t) = 0\}) = P(\{Z > z\} | \{X_1 > t\})$$

$$= P(\{X_1 > t + z\} | \{X_1 > t\}) \stackrel{\text{memorylessness}}{=} P(\{X_1 > z\}) = e^{-\lambda z}$$



- (ii) $P(\{Z > z\} | \{N(t) = n\}, \{S_n = t\}) = ?$



$$\text{let } S_n = t \leq t.$$

$$\text{and } N(t) = n \geq 1$$

$$P(\{Z > z\} | \{N(t) = n\}, \{S_n = t\}) = P(\{X_{n+1} > z + t - t\} | \{X_{n+1} > t - t\}, \{S_n = t\})$$

$$\stackrel{(a)}{=} P(\{X_{n+1} > z + t - t\} | \{X_{n+1} > t - t\})$$

$$\stackrel{(b)}{=} P(\{X_{n+1} > z\}) = e^{-\lambda z}$$

- (a) follows from the fact that X_{n+1} is independent of S_n .
 (b) follows from memoryless property.

$$X_{n+1} = S_{n+1} - S_n$$

$$\Rightarrow Z = S_{n+1} - t = X_{n+1} + S_n - t$$

⑨ Order Statistics :- [Goal is to compute $f_{S_1, \dots, S_n} | f_{N(t)=n}$]

Let $\gamma_1, \gamma_2, \dots, \gamma_n$ be n i.i.d jointly continuous RVs.

$\gamma_{(1)}, \gamma_{(2)}, \gamma_{(3)}, \dots, \gamma_{(n)} \longrightarrow$ "order statistics" corresponding to $\gamma_1, \gamma_2, \dots, \gamma_n$.

where $\gamma_{(1)} = \min \{ \gamma_1, \gamma_2, \dots, \gamma_n \} \longrightarrow$ 1st order statistic

$\gamma_{(2)} =$ second min $\{ \gamma_1, \dots, \gamma_n \} \longrightarrow$ 2nd " "

$\vdots$
 $\gamma_{(n)} = \max \{ \gamma_1, \dots, \gamma_n \} \longrightarrow$ nth order statistic

then it follows that the ^{joint} density of the order statistics is:

$$f_{\gamma_{(1)}, \gamma_{(2)}, \dots, \gamma_{(n)}}(y_1, \dots, y_n) = \begin{cases} n! \prod_{i=1}^n f(y_i), & y_1 < y_2 < \dots < y_n \\ 0 & , \text{ o.w.} \end{cases}$$

Observe that if $\gamma_1, \dots, \gamma_n$ are each uniformly distributed over $(0, t)$, then the joint density of the order statistics is:

$$f_{\gamma_{(1)}, \dots, \gamma_{(n)}}(y_1, \dots, y_n) = \begin{cases} \frac{n!}{t^n}, & 0 < y_1 < y_2 < \dots < y_n < t \\ 0 & , \text{ o.w.} \end{cases}$$

⑩ Order Statistics & Poisson Process :

Given that $N(t) = n$, $t \in [0, \infty)$, the arrival times $\tilde{S}_1, \dots, \tilde{S}_n$ are identically distributed as the order statistics corresponding to n iid $\text{Unif}(0, t)$ RVs.

Exercise 5: Compute $f_{\tilde{S}_1, \tilde{S}_2, \dots, \tilde{S}_n | N(t) = n}$.

Soln: - let us fix $0 < t_1 < t_2 < \dots < t_{n+1} = t$ and let δ_i be small enough s.t. $t_i + \delta_i < t_{i+1}$ for $i \in [n]$.

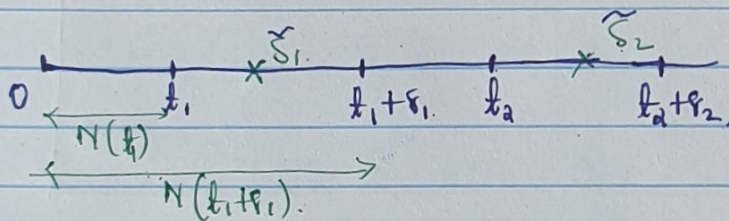
Then $P(t_i \leq \tilde{S}_i \leq t_i + \delta_i, \text{ for } i \in [n] | N(t) = n)$

$$= \frac{P(\{t_i \leq \tilde{S}_i \leq t_i + \delta_i, \text{ for } i \in [n]\} \cap \{N(t) = n\})}{P(\{N(t) = n\})} \quad (1)$$

Bayes rule $\uparrow$

$$= \frac{P(\{t_i \leq \tilde{S}_i \leq t_i + \delta_i, \text{ for } i \in [n]\}) \cdot P(\{N(t) = n\} | \{t_i \leq \tilde{S}_i \leq t_i + \delta_i\})}{P(\{N(t) = n\})} \quad (2)$$

$$(1) \quad P(t_1 \leq \tilde{S}_1 \leq t_1 + \delta_1) = P(\{N(t_1 + \delta_1) - N(t_1) = 1\}) \stackrel{\text{SIP}}{=} P(N(\delta_1) = 1) = \lambda \delta_1 e^{-\lambda \delta_1}$$



$$t_1 < t_2 \text{ and } t_1 + \delta_1 < t_2 \Rightarrow [t_1, t_1 + \delta_1] \text{ and } [t_2, t_2 + \delta_2] \text{ are disjoint intervals}$$

$$P(t_2 \leq \tilde{S}_2 \leq t_2 + \delta_2 | \{t_1 \leq \tilde{S}_1 \leq t_1 + \delta_1\})$$

$$= P(\{N(t_2 + \delta_2) - N(t_2) = 1\} | \{N(t_1 + \delta_1) - N(t_1) = 1\})$$

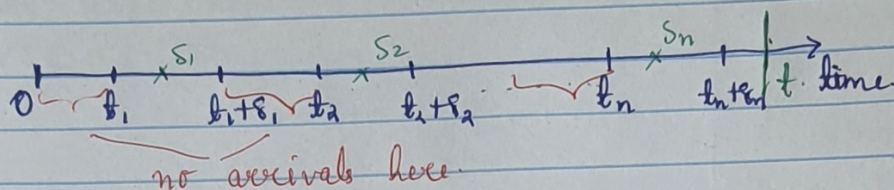
$$= P(\{N(t_2 + \delta_2) - N(t_2) = 1\})$$

IRP $\uparrow$

$$\stackrel{\text{SIP}}{=} P(N(\delta_2) = 1) = \lambda \delta_2 e^{-\lambda \delta_2}$$

Hence (1) $\stackrel{(2)}{=} (\lambda \delta_1 e^{-\lambda \delta_1}) (\lambda \delta_2 e^{-\lambda \delta_2}) \dots (\lambda \delta_n e^{-\lambda \delta_n})$

(2) $P(N(t)=n \mid s_i \in [t_i, t_i+\delta_i], \forall i \in [n]) = ?$
 here we know the arrival instants.



Hence (2) can be written as $P(N(t_1)=0, N(t_2)-N(t_1+\delta_1)=0, \dots, N(t_n)-N(t_n+\delta_n)=0)$

can be computed similar to exercise 2.

$$= e^{-\lambda t_1} \cdot e^{-\lambda(t_2 - (t_1 + \delta_1))} \cdot \dots \cdot e^{-\lambda(t - (t_n + \delta_n))}$$

$$= e^{-\lambda(t - \delta_1 - \delta_2 - \dots - \delta_n)}$$

Thus, $P(t_i \leq s_i \leq t_i + \delta_i, \text{ for } i \in [n] \mid N(t)=n)$

$$= \frac{(n! \cdot \delta_1 e^{-\lambda \delta_1} \cdot \delta_2 e^{-\lambda \delta_2} \cdot \dots \cdot \delta_n e^{-\lambda \delta_n}) \cdot e^{-\lambda(t - (t_n + \delta_n))}}{e^{-\lambda t} (\lambda t)^n / n!}$$

$$= \frac{n!}{t^n} \delta_1 \delta_2 \dots \delta_n$$

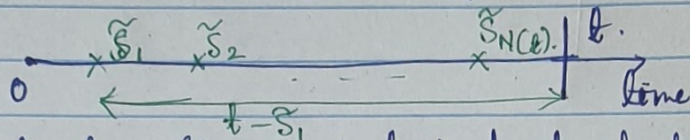
By dividing LHS & RHS by $\delta_1, \delta_2, \dots, \delta_n$ & letting $\delta_i \rightarrow 0$

$$\lim_{\substack{\delta_i \rightarrow 0, \\ \forall i \in [n]}} \frac{P(t_i \leq s_i \leq t_i + \delta_i, \forall i \in [n] \mid N(t)=n)}{\delta_1 \delta_2 \dots \delta_n} = \lim_{\substack{\delta_i \rightarrow 0 \\ \forall i \in [n]}} \frac{n!}{t^n}$$

$$\Rightarrow \boxed{f_{s_1, s_2, \dots, s_n \mid N(t)=n} = \frac{n!}{t^n}}$$

Exercise 6: Suppose travellers arrive at a bus stop according to a Poisson Process of rate λ . If the bus departs at time t , compute the expected sum of waiting times of arrivals in $(0, t)$.

Soln:-



$\tilde{s}_{N(t)}$ $\rightarrow$ denotes the arrival instant of last traveller before t

Waiting time of each traveller = $(t - \tilde{s}_i)$
 We have to compute $E \left[\sum_{i=1}^{N(t)} (t - \tilde{s}_i) \right] = E \left[E \left[\sum_{i=1}^{N(t)} (t - \tilde{s}_i) \mid N(t) \right] \right]$

$$E \left[\sum_{i=1}^{N(t)} (t - \tilde{s}_i) \mid N(t) = n \right] = E \left[\sum_{i=1}^n (t - \tilde{s}_i) \mid N(t) = n \right]$$

$$= nt - E \left[\sum_{i=1}^n \tilde{s}_i \mid N(t) = n \right]$$

[The mean is taken over $\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_n \mid N(t) = n$

$$= nt - n E[U]$$

$$= nt - n \frac{t}{2}$$

$$= \frac{nt}{2}$$

which has same distribution as iid uniform RVs on $(0, t)$

$$E \left[\sum_{i=1}^{N(t)} (t - \tilde{s}_i) \mid N(t) \right] = N(t) \cdot \left(\frac{t}{2} \right)$$

$$\therefore E \left[\sum_{i=1}^{N(t)} (t - \tilde{s}_i) \right] = E \left[N(t) \cdot \frac{t}{2} \right] = \frac{t}{2} \cdot \lambda t \quad (\because N(t) \sim \text{Poi}(\lambda t))$$

$$= \frac{\lambda t^2}{2}$$