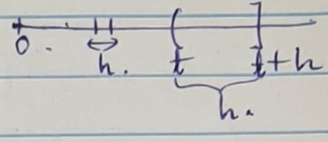
Alternate Defn. of Poisson Process : (Defn. 3)

A Poisson Process is a counting process $\{N(t), t \geq 0\}$ that satisfies the SIP, IIP and the following :

$$P(\{N(h) = 0\}) = 1 - \lambda h + o(h)$$
$$P(\{N(h) = 1\}) = \lambda h + o(h)$$
$$P(\{N(h) \geq 2\}) = o(h)$$



A func. is said to be $o(h)$ if $\lim_{h \to 0} \frac{f(h)}{h} = 0$

Interpretation :- ~~If~~ We look at number of arrivals in a very small interval. The prob. of having a single jump in the micro slot is very high and the prob. of having two or more jumps is very low ($o(h)$).

Proof that 2 $\Rightarrow$ 3

Let $N(t, s]$ $\rightarrow$ denote number of arrivals in $(t, s]$.

$$P(\{N(t, t+h] = 0\}) = P(\{N(t+h) - N(t) = 0\}) \underset{SIP}{=} P(\{N(h) - N(0) = 0\})$$

$$= P(\{N(h) = 0\}) = P(X_1 > h) = e^{-\lambda h} = \left(1 - \frac{\lambda h}{1!} + \frac{(\lambda h)^2}{2!} - \cdots\right)$$

$$= 1 - \lambda h + o(h).$$

$$P(\{N(t, t+h] = 1\}) = P(\{N(t+h) - N(t) = 1\}) = P(\{N(h) = 1\})$$
$$= \frac{e^{-\lambda h}(\lambda h)^1}{1!} = \lambda h(1 - \lambda h + o(h))$$
$$= \lambda h \underbrace{- \lambda^2 h^2 + o(h)}_{o(h)}$$
$$= \lambda h + o(h).$$

$$P(\{N(t, t+h) \geq 2\}) = 1 - P(\{N(t, t+h) = 0\}) - P(\{N(t, t+h) = 1\})$$

$$= 1 - (1 - \lambda h + o(h)) - (\lambda h + o(h)) = o(h).$$

Exercise 1 :- During working hours (9am – 5pm), customers arrive at a bank following a Poisson Process of arrival rate (20 customers per hour)

(a) What is the expected time of arrival of the fifth customer?

$$S_5 = \sum_{i=1}^{5} X_i \qquad X_i \sim \text{iid } \exp(20).$$

$$E[S_5] = E\left[\sum_{i=1}^{5} X_i\right] = 5.E[X_1] = 5.\frac{1}{20} = \frac{1}{4} \text{ hrs.}$$

(b) Given that only 10 customers arrived between 12pm and 1pm, what is the expected number of customers arriving between 1pm and 2pm?

$$N((12pm, 1pm]) = 10 \qquad P(N((1pm, 2pm]) = n \mid N((12pm, 1pm]) = 10)$$

$$\underset{IIP}{=} P(N((1pm, 2pm]) = n)$$

$$\underset{SIP}{=} P(N(1hr) = n) \rightarrow Poi(20 \text{ cust/hr} \times 1 \text{ hr})$$

$$\rightarrow Poi(20)$$

$$E[N((1pm, 2pm]) \mid N((12pm, 1pm]) = 10] = 20$$

TA Session 15

Superposition & Thinning

Let $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ be two independent homogeneous Poisson Processes of rates λ_1 and λ_2 respectively.

Theorem:— The merged process $\{N(t) = N_1(t) + N_2(t), t \geq 0\}$ is a Poisson process of rate $(\lambda_1 + \lambda_2)$.

* An arrival in $\{N(t), t \geq 0\}$ can come from the process $\{N_1(t), t \geq 0\}$, independently of any other arrival, w.p. $\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)$ and from $\{N_2(t), t \geq 0\}$ w.p. $\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)$

* Let $K \Rightarrow$ # arrivals in $\{N(t), t \geq 0\}$ coming from $\{N_1(t), t \geq 0\}$ before ~~any~~ first arrival comes from $\{N_2(t), t \geq 0\}$. Distribution of K:→

$$P(\{K = m\}) = \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^m \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right) \qquad (K+1) \sim Geo\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)$$

* Exercise 2 :— Show that the prob. of n events/arrivals from 1st process occur before m arrivals from 2nd process :—

$$\sum_{K=n}^{n+m-1} \binom{n+m-1}{k} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n+m-1-k}$$

Exercise 3 :— Flies and wasps land on your dinner plate in manner of independent poisson processes with respective intensities λ & μ. Show that arrivals of flying objects form a Poisson Process with intensity $(\lambda+\mu)$.

Soln.:— Let $\{F(t), t \geq 0\}$ & $\{W(t), t \geq 0\}$ be the incoming Poisson processes. $\{N(t) = F(t) + W(t), t \geq 0\}$ is the superposed process.

SIP & IIP follows from the respective properties of $F(t)$ & $W(t)$.

We use Defn. 3 to Show that the merged process, $\{N(t), t \geq 0\}$ is also a Poison Process of rate $(\lambda+\mu)$.

$$N(t, t+h] = N(t+h) - N(t)$$

$$P(\{N(t+h) - N(t) = 0\}) = P(\underbrace{\{F(t+h) - F(t) = 0\} \cap \{W(t+h) - W(t) = 0\}}_{\text{both processes have no arrivals}}).$$

$$\underset{\substack{\uparrow \\ \text{independence of } \{F(t), t\geq 0\} \text{ \& } \{W(t), t \geq 0\}}}{=} P(\{F(t+h) - F(t) = 0\}) \cdot P(\{W(t+h) - W(t) = 0\}).$$

$$= (1 - \lambda h + o(h)) \cdot (1 - \mu h + o(h)) \quad [\because F(t) \text{ \& } W(t) \text{ are PP}]$$

$$= 1 - (\lambda+\mu)h + o(h).$$

Similarly $P(\{N(t+h) - N(t) = 1\}) = P(\{F(t+h) - F(t) = 1\}) \cdot P(\{W(t+h) - W(t) = 0\})$
$+ P(\{F(t+h) - F(t) = 0\}) \cdot P(\{W(t+h) - W(t) = 1\})$

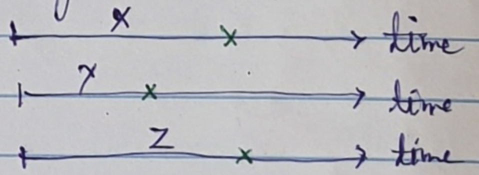
$$= (\lambda+\mu)h + o(h).$$

$P(\{N(t+h) - N(t) \geq 2\}) = \cancel{P(\{F(t+h) - F(t) = 1\}) \cdot P(\{W(t+h) - W(t) = 1\})}$
$\cancel{+ P(\{F(t+h) - F(t) = 0\}) \cdot P(\{}$

$$= 1 - P(\{N(t+h) - N(t) = 0\}) - P(\{N(t+h) - N(t) = 1\}) = o(h).$$

Definitions ① & ② can also be used to show that merged process is PP $(\lambda+\mu)$.

Example :- We have three lightbulbs. Each lightbulb is lit at time $t = 0$. The lifetimes of the 3 lightbulbs X, Y, Z; are independent RVs. $\sim \exp(\lambda)$. Find expected time until first burnout.



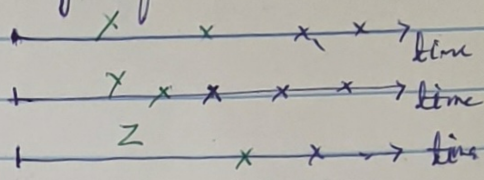
Approach 1 :-

$$E[\min\{X, Y, Z\}] = \int_0^\infty\int_0^\infty\int_0^\infty \min\{x, y, z\} \, \lambda e^{-\lambda x} \cdot \lambda e^{-\lambda y} \cdot \lambda e^{-\lambda z} \, dx\, dy\, dz$$

Approach 2 :- $P(\min\{X, Y, Z\} \geq t) = P(X \geq t, Y \geq t, Z \geq t) = e^{-\lambda t} e^{-\lambda t} e^{-\lambda t}$

$$= e^{-3\lambda t}$$

Approach 3:— Imagine that there is a PP running forever & X is the first arrival time.



So we interpret X, Y, Z are first arrival times in 3 independent Poison Processes.

We merge the 3 processes $\longrightarrow$ Poi (3λ).

The first arrival in the merged process will happen the first time one of the three processes will have arrival.

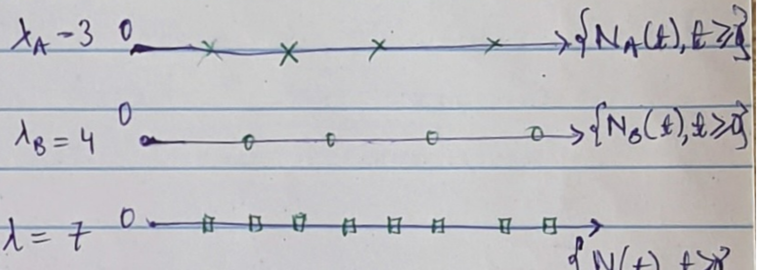
Thus 1st burnout time $\longrightarrow \min\{X, Y, Z\} \longrightarrow$ 1st arrival in the merged process is Exp(3λ).

Thus $E[\text{1st Burnout time}] = \left(\dfrac{1}{3\lambda}\right)$.

Exercise 4: A service station handles jobs of two types A and B. Arrivals of two job types are independent Poisson processes with parameters $\lambda_A = 3$, $\lambda_B = 4$ per min. respectively.

(a) What is mean, variance and PMF of the total number of jobs that arrive within a 3-min interval?

Soln.



$N(3) \longrightarrow$ no. of arrivals in 3 min interval.

$E[N(3)] = \lambda . 3$

$= 7.3 = 21$ $\quad$ [$\because \{N(t), t \geq 0\}$ is Poisson Process, $N(t)$ has Poisson distribution].

$Var[N(3)] = \lambda t = 21.$ $\qquad P_N(K) = \dfrac{(21)^K e^{-21}}{K!}$ $\quad K = 0, 1, 2, \ldots$

(b) During a 10-min interval, exactly 10 new jobs arrive. What is the prob. that exactly three of them are of type A?

Soln. Prob. that an arrival is of type A,

$$= P(\{X_A \leq X_B\}) = \frac{\lambda_A}{\lambda_A + \lambda_B} = \left(\frac{3}{7}\right).$$

Each of the 10 jobs has probability $\left(\frac{3}{7}\right)$ of being type A, independent of others.

$$\binom{10}{3} \cdot \left(\frac{3}{7}\right)^3 \left(\frac{4}{7}\right)^7$$

(c) At time 0, no job is present in the service station. What is the PMF of the no. of type B jobs that arrive in the future, but before the first type A arrival?

Soln :- Let there are K arrivals of type B before the 1st type A arrival.

$$p_K(k) = P_r(K = k) = \left(1 - \frac{3}{7}\right)^k \left(\frac{3}{7}\right) = \left(\frac{4}{7}\right)^k \left(\frac{3}{7}\right) \quad k = 0, 1, 2, \ldots$$

$$K+1 \sim \text{geo}\left(\frac{3}{7}\right)$$

Exercise 5 :- Consider lighting of a room process, where the room is lighted by type-A bulbs or type-B bulbs independently and equally likely immediately upon the failure of previously installed bulb. The lifetimes of the bulbs A and B are exponentially distributed random variables with rates 1 and 3 respectively.

(a) Find the expected time until first failure.

Soln :- $T \rightarrow$ time till first failure.

$$E[T] = E[T|A] \cdot P(A) + E[T|B] \cdot P(B)$$

↑ Total Expectation

where $A \rightarrow$ event that first bulb is of type A.

$B \rightarrow$ " " " type B.

$$= 1 \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} = \frac{2}{3}$$

$\Big\{$ $[T|A]$ is the lifetime of bulb A. $\sim \exp(\lambda_1)$. $\lambda_1 = 1$

Hence $E[T|A] = \frac{1}{\lambda_1} = \frac{1}{1} = 1$ Similarly $E[T|B] = \frac{1}{\lambda_2} = \frac{1}{3}$ $\Big\}$

(b) Find the prob. that there are no bulb failures before time t.

Soln :- Using total prob. theorem and exponential distributions for bulbs of two types, we get

$$P(\{T \geq t\}) = P(\{T \geq t\} \mid A) \cdot P(A) + P(\{T \geq t\} \mid B) \cdot P(B)$$

$$= P(\{X_1^A \geq t\} \mid A) \cdot P(A) + P(\{X_1^B \geq t\} \mid B) \cdot P(B)$$ where $X_1^A \rightarrow$ 1st type A bulb ~~failure~~ ~~instant~~ lifetime.

$$= \frac{1}{2} \cdot e^{-t} + \frac{1}{2} e^{-3t}.$$

$X_1^B \rightarrow$ 1st type B bulb lifetime

(c) Given there are no failures until time t, determine the conditional probability that the first bulb used is a type-A bulb.

Soln-

$$P(A \mid \{T \geq t\}) \underset{\text{Bayes}}{=} \frac{P(\{T \geq t\} \mid A) \cdot P(A)}{P(\{T \geq t\})} = \frac{\frac{1}{2} e^{-t}}{\frac{1}{2} e^{-t} + \frac{1}{2} e^{-3t}} = \frac{1}{1 + e^{-2t}}.$$

(d) Determine the prob. that the total period of illumination provided by the first two type-B bulbs is longer than that provided by the first type-A bulb.

Soln: $X_1^A \rightarrow$ period of illumination of the first type-A bulb.
X_1^B, $X_2^B \rightarrow$ " " " provided by the first and second type-B bulb, respectively.

Approach 1 :- (We do not look at it as a Poisson Process).

$$P(X_1^A < X_1^B + X_2^B) = P(X_1^A < Y^B)$$ $Y^B \rightarrow$ total period of illumination provided by the 1st 2 type-B bulbs.

$$f_{Y^B}(y) = 9y\, e^{-3y}, \quad y \geq 0.$$

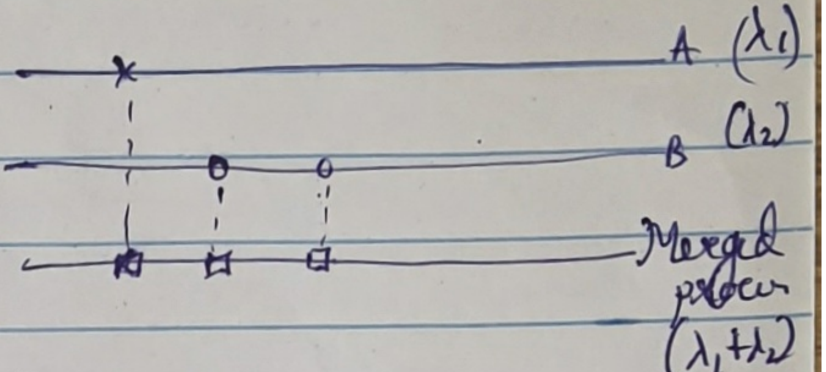
$$f_{X_1^A}(x) = e^{-x}, \quad x \geq 0$$

$$P(X_1^A < Y^B) = P(Y^B > X_1^A) = \int_{y=0}^{\infty} \int_{x=0}^{y} e^{-x}\, dx \cdot 9 e^{-3y} y\, dy$$

$$= \int_0^\infty (1-e^{-y})\, 9 e^{-3y}\, y \, dy \;=\; 9\left(\int_0^\infty y e^{-3y}\, dy - \int_0^\infty y e^{-4y}\, dy\right)$$

$$= \left(\frac{7}{16}\right).$$

Alternate approach using Poisson Process :-



Cases of interest :- 1st arrival from A followed by B and B A B.

$$P(X_1^B + X_2^B > X_1^A) = P(A) + P(B).P(A)$$

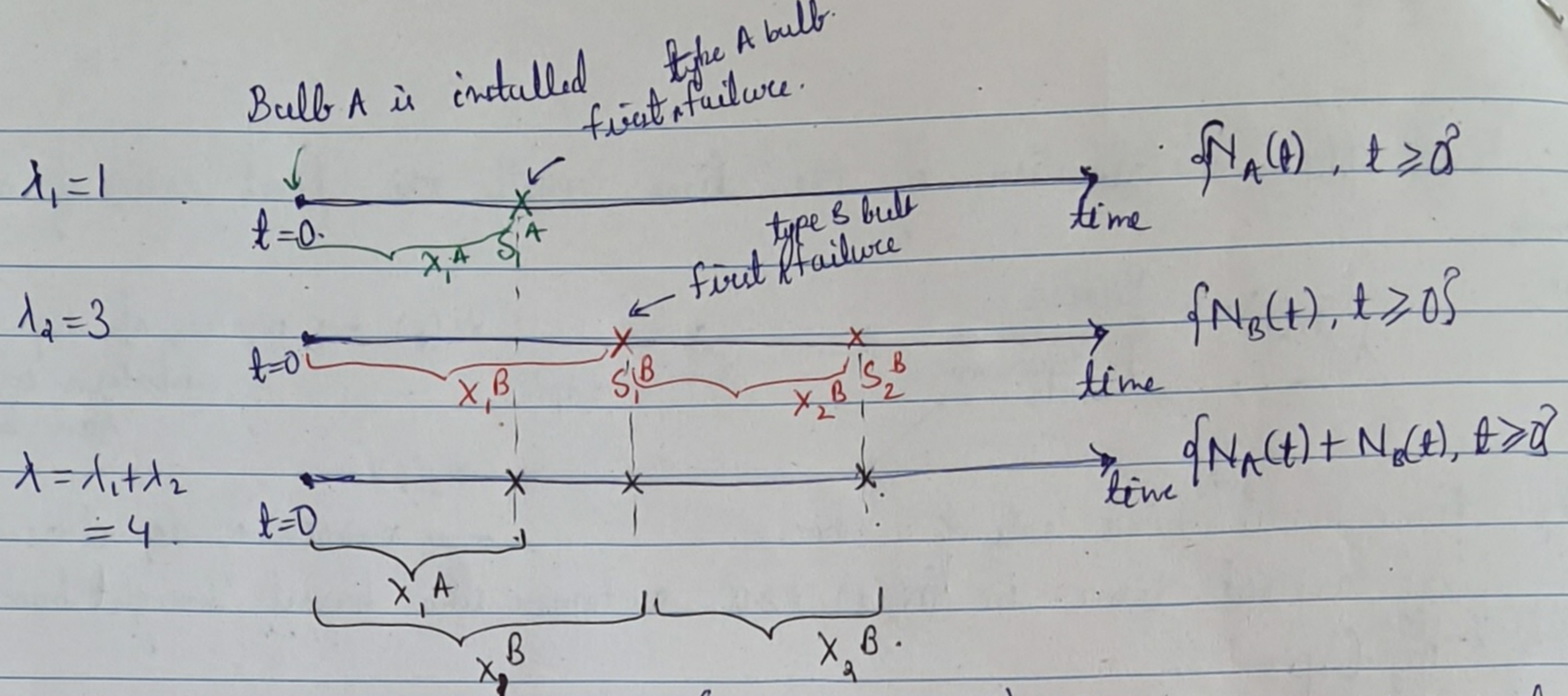
$$= \left(\frac{1}{4}\right) + \left(\frac{1}{4}\right) \times \left(\frac{3}{4}\right) = \left(\frac{7}{16}\right).$$

Alternate approach :—

We embed the failure times into appropriate Poisson process. Let process $\{N_A(t), t \geq 0\}$ represent the type A bulb process where arrival instants, S_i^A correspond to failure times of bulb A and interarrival times $X_i^A \sim \exp(\lambda_1)$ represent the lifetimes of bulb A. Process $\{N_B(t), t \geq 0\}$ is defined in ~~a~~ similar ~~fas~~ manner.

We look at the combined process $\{N(t), t \geq 0\}$

$$= \{N_A(t) + N_B(t), t \geq 0\}.$$



We are interested in $P(X_1^A < X_1^B + X_2^B)$ where $X_1^A \to$ lifetime of 1st A bulb

$X_1^B \to$ lifetime of 1st B "

$X_2^B \to$ " " 2nd " ".

The cases which satisfy ~~our~~ the condition :— In merged process, 1st arrival is type A bulb

or. 1st arrival is type B bulb followed by type A bulb.

Exercise 6 : A fisherman catches fish acc. to a Poisson process with rate $\lambda = 0.6$ per hour. The fisherman will keep fishing for two hours. If he has caught at least one fish, he quits. Otherwise, he continues until he catches at least one fish.

(a) Find the probability that he stays for more than two hours.

Soln :-

$$P(\underbrace{N(2)=0}_{\text{no arrivals in 2 hrs.}}) = e^{-0.6\times 2} = e^{-1.2}. \quad \text{or.} \quad P(\underbrace{X_1 > 2}_{\text{1st arrival comes after 2hrs.}}) = \int_2^\infty f_{X_1}(x)\,dx$$

$$= \int_2^\infty 0.6\, e^{-0.6x}\,dx$$

$$= e^{-0.6\cdot 2}.$$

(b) Find the prob. that ~~he stays for more than two hours.~~ total time he spends fishing is between two and five hours.

Soln :-

$$P(\{2 \leq T \leq 5\}) = P(\{0 \text{ arrival } \cancel{\text{betn.}} \text{ in } (0,2]\} \cap \{\text{atleast one arrival in } (2,5]\})$$

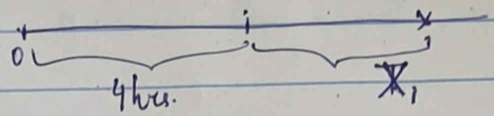
$$= P(\{N(2)=0\}) \cdot P(\{N(5)-N(2) \geq 1\})$$

$$= P(\{N(2)=0\}) \left[1 - P(N(3) < 1)\right] = P(\{N(2)=0\}) \cdot \left[1 - P(N(3)=0)\right]$$

$= e^{-1.2} \cdot \left(1 - \frac{(0.6 \times 3)^0 e^{-0.6\times 3}}{0!}\right) = e^{-1.2}\left(1 - e^{-1.8}\right). \quad = 0.251.$

or. $\{2 \leq T \leq 5\} = \int_2^5 f_T(t)\,dt = \int_2^5 0.6 e^{-0.6t}\,dt = 0.251$

(c) Find the expected total fishing time, given that he has been fishing for four hours.



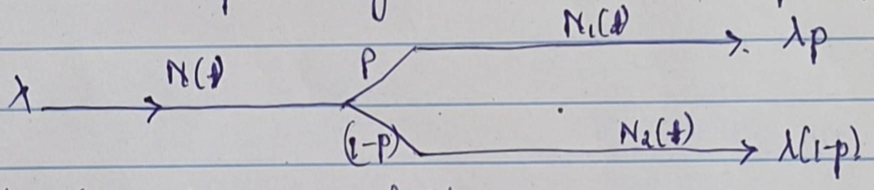
$E[\text{Total fishing time}] = E[4 + X_1]$

Due to memorylessness of the PP, the future time is exponential with mean $\left(\frac{1}{\lambda}\right)$.

Hence $E[4 + X_1] = 4 + E[X_1] = 4 + \frac{1}{0.6} = 5.667.$

Poisson Splitting/Thinning :-

Consider a homogeneous Poisson Process $\{N(t), t \geq 0\}$ of rate $\lambda > 0$. A splitting of the process into two independent processes $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ is carried out as:
each arrival in $N(t)$ is independently labelled $N_1(t)$ w.p. p & labelled $N_2(t)$ w.p. $(1-p)$.



We can prove that the following hold :-

(i) $\{N_1(t), t \geq 0\}$ & $\{N_2(t), t \geq 0\}$ are PPs of rates λp & $\lambda(1-p)$ respectively.

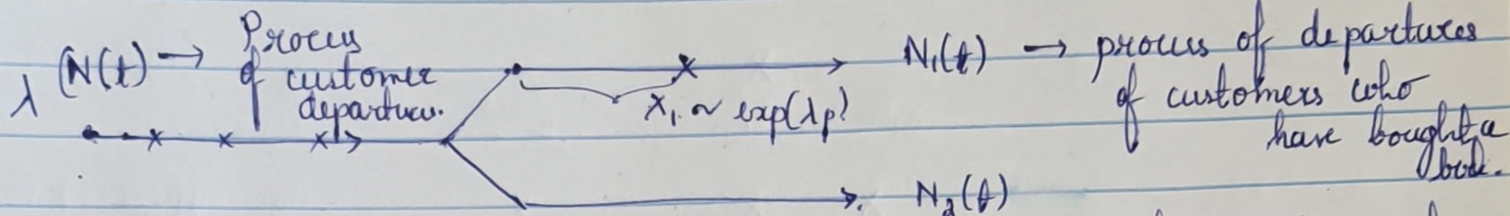
(ii) They are independent Poisson processes.

~~Example~~ Exercise 7 :-

Customers depart from a bookstore acc. to a Poisson Process with rate λ per hr. Each customer buys a book w.p. p independent of everything else.

(a) Find the distribution of the time until the first sale of a book.

Soln



The time until first sale of a book.
= first arrival time in $\{N_1(t), t \geq 0\}$
distribution $\stackrel{=}{}$ $exp(\lambda p)$

(b) Find the prob. that no books are sold during a particular hr.

Soln

P (no customers in the split PP $N_1(t)$ during an hr)

$$= P(\{N_1(t) = 0\}) = P(\{N_1(1) = 0\}) = \frac{(\lambda p t)^0 e^{-\lambda p t}}{0!} = e^{-\lambda p (1)}$$
$$= e^{-\lambda p}.$$

(c) ~~Find the~~ The expected number of customers in the split Poisson process during an hour.

= expected number of customers who buy book during a particular hour

$$= E[N_1(t)] = \lambda p(1) = \lambda p.$$